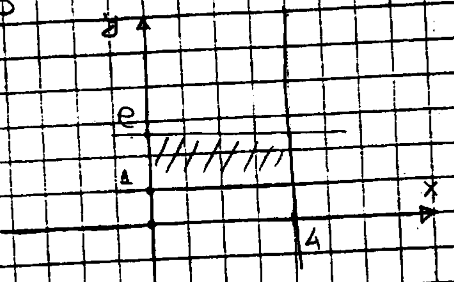
6.2. Двојни интеграл по датој области

Израчунати следеће двојне интеграле ако је дата област интеграције D:

1. $\iint\limits_D x\ln y\,dx\,dy \qquad D=\{(x,y)\mid 0\le x\le 4 \wedge 1\le y\le e\}$



$$I=\int_0^4 dx\int_1^e x\ln y\,dy$$

$$I=\int_0^4 dx\cdot x\int_1^e \ln y\,dy=\begin{bmatrix} u=\ln y & du=\frac{dy}{y}\\ dv=dy & v=y\end{bmatrix}$$

$$I=\int_0^4 x\,dx\left(y\ln y\Big|_1^e-\int_1^e dy\right)$$

$$I=\int_0^4 x\,dx\left(e-y\Big|_1^e\right)=\int_0^4 x\,dx\,(\cancel{e}-\cancel{e}+1)$$

$$I=\frac{x^2}{2}\Big|_0^4=8$$

2. $\iint\limits_D \frac{x^2}{1+y^2}\,dx\,dy \qquad D=\{(x,y)\mid 0\le x\le 4 \wedge -1\le y\le 1\}$

$\operatorname{tg}x=1 \quad x=\frac{\pi}{4}$

$$I=\int_0^4 x^2dx\int_{-1}^1\frac{dy}{1+y^2}=\int_0^4 x^2dx\left[\operatorname{arctg}y\Big|_{-1}^1\right]$$

$$I=\int_0^4 x^2dx\left[\operatorname{arctg}1-\operatorname{arctg}(-1)\right]=\int_0^4 x^2dx\left(\frac{\pi}{4}+\frac{\pi}{4}\right)=\frac{\pi}{2}\int_0^4 x^2dx=\frac{\pi}{2}\frac{x^3}{3}=\frac{32\pi}{3}$$

3. $\iint\limits_D (\cos^2x+\sin^2y)\,dx\,dy \qquad D=\left\{(x,y)\mid \frac{\pi}{4}\le x\le\frac{\pi}{2} \wedge \frac{\pi}{3}\le y\le\frac{2\pi}{3}\right\}$

$$\int_{\pi/4}^{\pi/2}dx\int_{\pi/3}^{2\pi/3}(\cos^2x+\sin^2y)\,dy=\int_{\pi/4}^{\pi/2}dx\left[(\cos^2x\cdot y)\Big|_{\pi/3}^{2\pi/3}+\int_{\pi/3}^{2\pi/3}\frac{1-\cos2y}{2}dy\right]$$

$$\int_{\pi/4}^{\pi/2}dx\left(\frac{\pi}{3}\cos^2x+\frac{y}{2}-\frac{\sin2y}{4}\Big|_{\pi/3}^{2\pi/3}\right)=\int_{\pi/4}^{\pi/2}\left[\frac{\pi}{3}\cos^2x+\frac{\pi}{6}-\frac{1}{4}\left(\sin\frac{4\pi}{3}-\sin\frac{2\pi}{3}\right)\right]dx$$

$$=\int_{\pi/4}^{\pi/2}\left(\frac{\pi}{3}\cos^2x+\frac{\pi}{6}+\frac{1}{2}\sin\frac{\pi}{3}\right)dx=\int\left(\frac{\pi}{3}\cos^2x+\frac{\pi}{6}+\frac{\sqrt3}{4}\right)dx$$

$$=\frac{\pi}{3}\int_{\pi/4}^{\pi/2}\frac{1+\cos2x}{2}dx+\left(\frac{\pi}{6}+\frac{\sqrt3}{4}\right)\frac{\pi}{4}=\frac{\pi}{3}\left[\left(\frac{1}{2}\cdot\frac{\pi}{4}\right)+\frac{1}{4}\sin2x\Big|_{\pi/4}^{\pi/2}\right]+$$

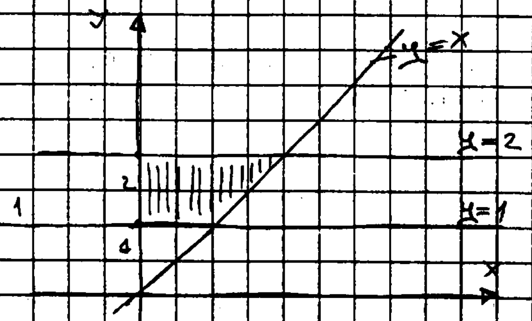
4. $\iint\limits_D e^{x+\sin y}\cos y\,dx\,dy \qquad D=\{(x,y)\mid 0\le x\le \pi \wedge 0\le y\le \frac{\pi}{2}\}$

$$=\int_0^{\pi}dx\int_0^{\pi/2}e^x\cdot e^{\sin y}\cos y\,dy = \qquad \sin y=t \quad \cos y\,dy=dt$$

$$=\int_0^{\pi}dx\, e^x\cdot e^{\sin y}\Big|_0^{\pi/2}=\int_0^{\pi}e^x dx\,(e^{\sin\pi/2}-e^{\sin 0})=\int_0^{\pi}e^x(e^1-e^0)\,dx$$

$$(e-1)\int_0^{\pi}e^x dx=(e-1)e^x\Big|_0^{\pi}=(e-1)(e^{\pi}-e^0)=(e-1)(e^{\pi}-1)$$

5. $\iint\limits_D (x^2+y^2)\,dx\,dy$, D – огр. правама $x=0$, $y=x$, $y=1$, $y=2$



$1\le y\le 2$

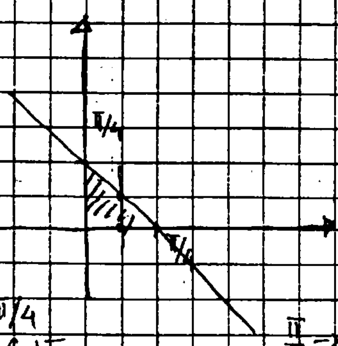
$0\le x\le y$

$$I=\int_1^2 dy\int_2^y (x^2+y^2)\,dx=\int_1^2 dy\left[\frac{x^3}{3}+y^2x\right]\Big|_0^y=\int_1^2\left[\frac{y^3}{3}+y^3\right]dy$$

$$=\frac{y^4}{12}+\frac{y^4}{4}\Big|_1^2=\frac{y^4+3y^4}{12}\Big|_1^2=\frac{4y^4}{12}\Big|_1^2=\frac{1}{3}y^4\Big|_1^2=\frac{1}{3}\cdot 15=5$$

6. $\iint\limits_D (\cos 2x+\sin y)\,dx\,dy \qquad D:\ x=0,\ y=0 \quad 4x+4y-\pi=0$

$x+y=\frac{\pi}{4} \qquad y=\frac{\pi}{4}-x$



$0\le x\le \pi/4$

$0\le y\le \frac{\pi}{4}-x$

$$I=\int_0^{\pi/4}dx\int_0^{\frac{\pi}{4}-x}(\cos 2x+\sin y)\,dy$$

$$I=\int_0^{\pi/4}\left[\cos 2x\cdot y\Big|_0^{\frac{\pi}{4}-x}-\cos y\Big|_0^{\pi/4-x}\right]dy=$$

$$I = \int_0^{\pi/4} \left(\frac{\pi}{4} - x\right)\cos 2x - \left[\cos\left(\frac{\pi}{4} - x\right) - \cos 0\right] dx$$

$$\cos\left(\frac{\pi}{4} - x\right) = \cos\frac{\pi}{4}\cos x + \sin\frac{\pi}{4}\sin x = \frac{1}{\sqrt{2}}(\cos x + \sin x)$$

$$I = \int_0^{\pi/4} \left[\frac{\pi}{4}\cos 2x - x\cos 2x - \frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x - 1\right] dx$$

$x = u \quad du = dx$

$dv = \cos 2x \, dx$

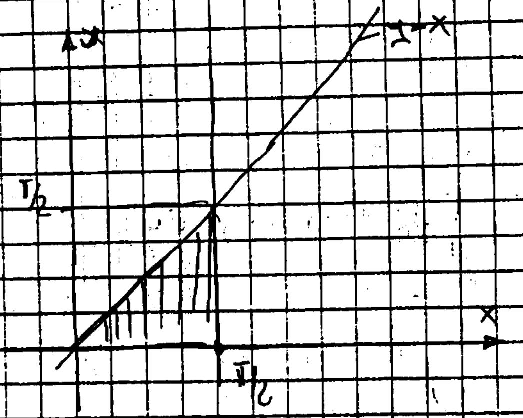
$v = \frac{\sin 2x}{2}$

$$I = \frac{\pi}{8}\sin 2x\Big|_0^{\pi/4} - \frac{1}{\sqrt{2}}\sin x\Big|_0^{\pi/4} + \frac{1}{\sqrt{2}}\cos x\Big|_0^{\pi/4} - x\Big|_0^{\pi/4} - \int_0^{\pi/4} x\cos 2x\,dx$$

$$I = \frac{\pi}{8} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - 1 - \frac{\pi}{4} - \left[\frac{x\sin 2x}{2}\Big|_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4}\sin 2x\,dx\right]$$

$$I = -\frac{\pi}{8} - 1 - \left[\frac{\pi}{8} + \frac{1}{4}\cos 2x\Big|_0^{\pi/4}\right] = -\frac{\pi}{8} - 1 - \frac{\pi}{8} + \frac{1}{4}[-1] = -\frac{\pi}{4} - \frac{5}{4} = \frac{-\pi-5}{4}$$

17. $\iint_D \sin(x+y)\,dx\,dy \qquad y = 0, \quad x = \frac{\pi}{2}, \quad x = y$



$0 \le x \le \pi/2$

$0 \le y \le x$

$\sin(x+y) = \sin x\cos y + \sin y\cos x$

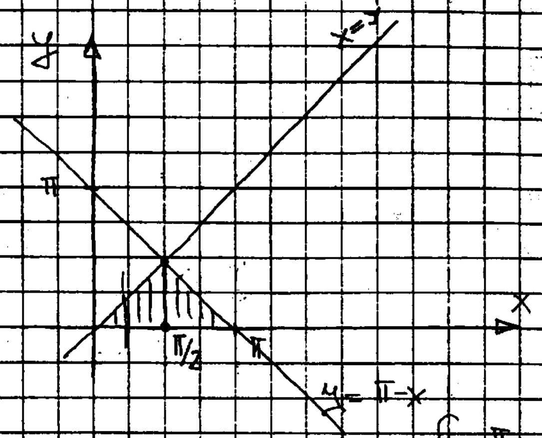
$$\int_0^{\pi/2} dx \int_0^x \sin(x+y)\,dy = \int_0^{\pi/2} dx \int_0^x (\sin x\cos y + \sin y\cos x)\,dy$$

$$= \int_0^{\pi/2} dx \left[\sin x \sin y - \cos y\cos x\right]\Big|_0^x = \int_0^{\pi/2} (\sin x\sin x - \cos x\cos x)\,dx$$

$$= -\int_0^{\pi/2} (\cos x\cos x - \sin x\sin x)\,dx = -\int_0^{\pi/2} \cos 2x = -\frac{\sin 2x}{2} = 0$$

8) $\iint_D \sin(x+y)\,dx\,dy$ $\qquad y=0,\ x+y=\pi,\ x=y$

$y=\pi-x$ $\quad \begin{array}{c|c|c} x & 0 & \pi \\ y & \pi & 0 \end{array}$



$y=x \cap \pi-x=y$

$x=\pi-x \Rightarrow x=\frac{\pi}{2}$

$D_1: \begin{cases} 0 \le x \le \pi/2 \\ 0 \le y \le x \end{cases} \qquad D_2: \begin{cases} \frac{\pi}{2} \le x \le \pi \\ 0 \le y \le \pi - x \end{cases}$

$$\iint_D = \iint_{D_1} + \iint_{D_2} = \int_0^{\pi/2} dx \int_0^{x} \sin(x+y)\,dy + \int_{\pi/2}^{\pi} dx \int_0^{\pi-x} \sin(x+y)\,dy$$

* $\sin(x+y) = \sin x\cos y + \sin y\cos x$

$$I = \int_0^{\pi/2} dx \int_0^{x} \sin x\cos y + \sin y \cos x \, dy + \int_{\pi/2}^{\pi} dx \int_0^{\pi-x} \sin(x+y)\,dy$$

$$I = \int_0^{\pi/2} dx\,(+\sin x\sin y - \cos x\cos y)\Big|_0^{x} + \int_{\pi/2}^{\pi} dx\,(\sin x \sin y - \cos x\cos y)$$

$$I = -\int_0^{\pi/2} dx\,[\cos(x+y)]\Big|_0^{x} - \int_{\pi/2}^{\pi} dx\,(\cos(x+y))\Big|_0^{\pi-x}$$

$$I = -\int_0^{\pi/2} (\cos 2x - \cos x)\,dx - \int_{\pi/2}^{\pi} (\underbrace{\cos\pi}_{-1} - \cos x)\,dx$$

$$I = \int_0^{\pi/2} \cos x\,dx - \int_0^{\pi/2} \cos 2x\,dx + x\Big|_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \cos x\,dx$$

$$I = \sin x\Big|_0^{\pi/2} - \frac{\sin 2x}{2}\Big|_0^{\pi/2} + \frac{\pi}{2} + \sin x\Big|_{\pi/2}^{\pi} = 1 + \frac{\pi}{2} - 1 = \frac{\pi}{2}$$

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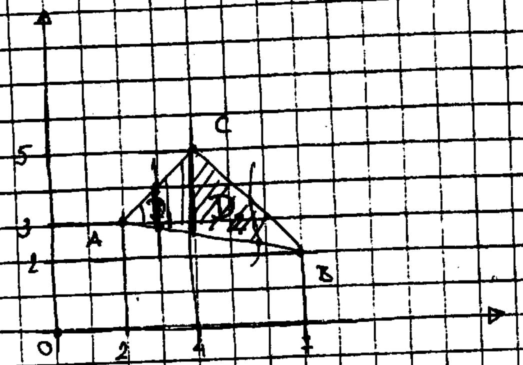
9. $\iint\limits_D x\,dx\,dy$ $\quad A(2,3),\ B(7,2),\ C(4,5)$



$\overline{AB}$: $y - y_A = \dfrac{y_B - y_A}{x_B - x_A}(x - x_A)$

$y-3 = \dfrac{2-3}{7-2}(x-2)$

$y-3 = -\dfrac{1}{5}(x-2)$

$y = -\dfrac{x}{5} + \dfrac{2}{5} + 3 = -\dfrac{x}{5} + \dfrac{2+15}{5}$

$\boxed{y_{AB} = -\dfrac{x}{5} + \dfrac{17}{5}}$

$\overline{AC}$: $y-3 = \dfrac{5-3}{4-2}(x-2)$

$y-3 = x-2 \quad \boxed{y = x+1}$

$\overline{CB}$: $y-5 = \dfrac{2-5}{7-4}(x-4) \Rightarrow y-5 = -x+4 \quad \boxed{y = 9-x}$

$D_1 = \begin{cases} 2 \le x \le 4 \\ \dfrac{17-x}{5} \le y \le x+1 \end{cases} \qquad D_2 = \begin{cases} 4 \le x \le 7 \\ \dfrac{17-x}{5} \le y \le 9-x \end{cases}$

$I = \int_2^4 x\,dx \int_{\frac{17-x}{5}}^{x+1} dy + \int_4^7 x\,dx \int_{\frac{17-x}{5}}^{9-x} dy = \int_2^4 x\,dx\left(x+1 - \dfrac{17-x}{5}\right) + \int_4^7 x\,dx\left(9-x-\dfrac{17-x}{5}\right)$

$= \int_2^4 \left[\dfrac{5x+5-17+x}{5}\right] x\,dx + \int_4^7 \left[\dfrac{45-5x-17+x}{5}\right] x\,dx = \int_2^4 \left(\dfrac{6x^2-12x}{5}\right)dx + \int_4^7 \left(\dfrac{28x-4x^2}{5}\right)dx$

$= \dfrac{1}{5}\left(2x^3 - 6x\right)\Big|_2^4 + \dfrac{1}{5}\left(14x - \dfrac{4}{3}x^3\right)\Big|_4^7$

$= \dfrac{1}{5}\left[2\cdot 4^3 - 6\cdot 4 - 2\cdot 2^3 + 6\cdot 2 + 14\cdot 7 - \dfrac{4}{3}\cdot 7^3 - 14\cdot 4 + \dfrac{4}{3}\cdot 4^3\right] = \dots$

10. $\iint\limits_D (2x+3y+1)\,dx\,dy$ $\quad A(-1,-1),\ B(2,0),\ C(1,1)$

$AB \Rightarrow y+1 = \dfrac{0+1}{2+1}(x+1) \Rightarrow$

$y = \dfrac{1}{3}x + \dfrac{1}{3} - 1 = \dfrac{x-2}{3}$

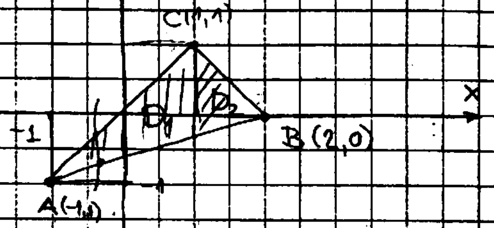
$\underline{AC}$: $y+1 = \dfrac{1+1}{1+1}(x+1) \Rightarrow y+1 = x+1$

$y = x$

$\underline{CB}$: $y-1 = \dfrac{0-1}{2-1}(x-1) \Rightarrow y-1 = -\dfrac{1}{2}(x-1) \Rightarrow y = -\dfrac{1}{2}x + \dfrac{1}{2} + 1$

$y = -\dfrac{1}{2}x + \dfrac{3}{2}$

$\underline{AB}$: $y = \frac{x-2}{3}$ $\quad$ $\underline{AC}$: $y = x$ $\quad$ $\underline{CB}$: $y = \frac{3-x}{2}$



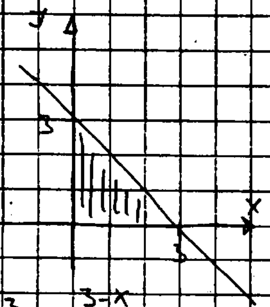
D_1: $\begin{cases} -1 \le x \le 1 \\ \frac{x-2}{3} \le y \le x \end{cases}$ $\quad$ D_2: $\begin{cases} 1 \le x \le 2 \\ \frac{x-2}{3} \le y \le \frac{3-x}{2} \end{cases}$

$$I = \int_{-1}^{1} dx \int_{\frac{x-2}{3}}^{x} (2x+3y+1)\,dy + \int_{1}^{2} dx \int_{\frac{x-2}{3}}^{\frac{3-x}{2}} (2x+3y+1)\,dy$$

$$I = \int_{-1}^{1} \left(2xy + \frac{3}{2}y^2 + y\right)\Big|_{\frac{x-2}{3}}^{x} dx + \int_{1}^{2} \left(2xy + \frac{3}{2}y^2 + y\right)\Big|_{\frac{x-2}{3}}^{\frac{3-x}{2}} dx$$

$$I = \int_{-1}^{1} \left(2x^2 + \frac{3}{2}x^2 + x - \frac{2x}{3}(x-2) - \frac{1}{2}(x-2)^2 + \frac{2-x}{3}\right) dx + \ldots = \ldots$$

$\boxed{11.}$ $\iint_D (2x+y)\,dx\,dy$ $\quad$ D: $x+y=3$, x-axa, Oy



D: $\begin{cases} 0 \le x \le 3 \\ 0 \le y \le 3-x \end{cases}$

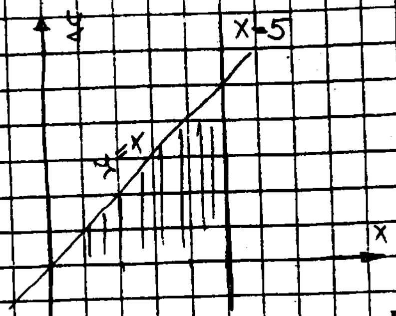
$$I = \int_0^3 dx \int_0^{3-x} (2x+y)\,dy = \int_0^3 dx \left(2xy + \frac{y^2}{2}\right)\Big|_0^{3-x}$$

$$= \int_0^3 \left(2x(3-x) + \frac{(3-x)^2}{2}\right) dx = \int_0^3 \left(6x - 2x^2 + \frac{9}{2} - 3x + \frac{x^2}{4}\right) dx$$

$$= \int_0^3 \left(\frac{-7x^2}{4} + 3x + \frac{9}{2}\right) dx = -\frac{7x^3}{12} + \frac{3}{2}x^2 + \frac{9}{2}x \Big|_0^3 = -\frac{7}{\cancel{12}\,4}\cdot 3^{\cancel{3}\,2} + \frac{3}{2}\cdot 3^2 + \frac{9}{2}\cdot 3$$

$$= -\frac{7}{4}\cdot 9 + \frac{27}{2} + \frac{27}{2} = -\frac{63}{4} + \frac{54}{2} = -\frac{63}{4} + 27$$

12. $\iint\limits_D (x+2y)\,dx\,dy$ $x=5,\ y=0,\ y=x$

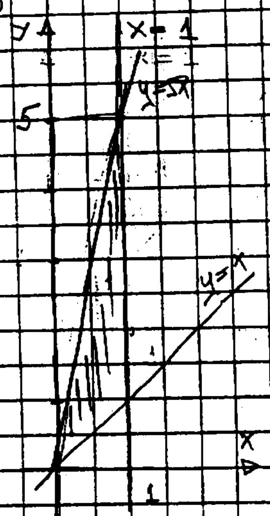


$$D:\begin{cases} 0\le x\le 5 \\ 0\le y\le x \end{cases}$$

$$I=\int_0^5 dx\int_0^x (x+2y)\,dy$$

$$I=\int_0^5 dx\,(xy+y^2)\Big|_0^x=\int_0^5 (x^2+x^2)\,dx=\frac{2}{3}x^3\Big|_0^5=\frac{2}{3}\cdot 125=\frac{250}{3}$$

13. $\iint\limits_D (x+6y)\,dx\,dy$ $y=x,\ y=5x,\ x=1$



$$D:\begin{cases} 0\le x\le 1 \\ x\le y\le 5x \end{cases}$$

$$I=\int_0^1 dx\int_x^{5x}(x+6y)\,dy$$

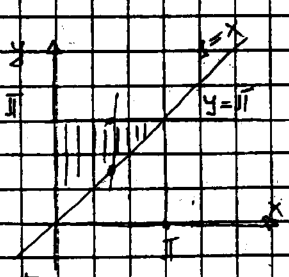
$$=\int_0^1 dx\,(xy+3y^2)\Big|_x^{5x}$$

$$=\int_0^1 (5x^2+75x^2-x^2-3x^2)\,dx=\int_0^1 76x^2\,dx=\frac{76}{3}x^3=\frac{76}{3}$$

14. $\iint\limits_D \cos(x+y)\,dx\,dy$

$$D:\begin{cases} 0\le x\le \pi \\ x\le y\le \pi \end{cases}$$

D: $x=0,\ x=y,\ y=\pi$



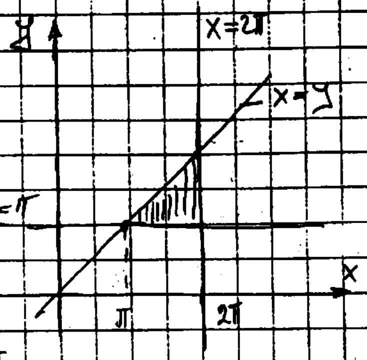
$$I=\int_0^\pi dx\int_x^\pi \cos(x+y)\,dy=\left[\ x+y=t,\ dy=dt\ \right]$$

$$I=\int_0^\pi dx\,\sin(x+y)\Big|_x^\pi=\int_0^\pi\left[\sin(x+\pi)-\sin 2x\right]dx$$

$$=-\cos(x+\pi)\Big|_0^\pi+\frac{\cos 2x}{2}\Big|_0^\pi=-\left[\cos 2\pi-\cos\pi\right]+\frac{1}{2}(\cos 2\pi-\cos 0)$$

$$=-(1+1)=-2$$

115. $\iint_D \cos(x+y)\,dx\,dy$ $\quad x=2\pi,\ x=y,\ y=\pi$



$D: \begin{cases} \pi \le x \le 2\pi \\ \pi \le y \le x \end{cases}$

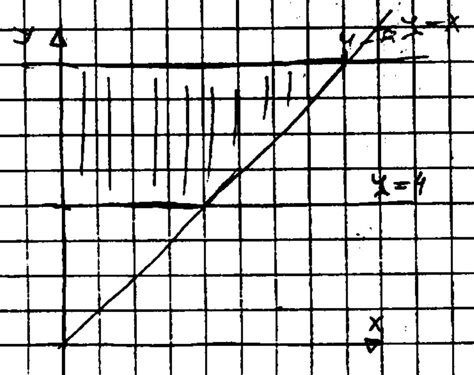
$I = \int_{\pi}^{2\pi} dx \int_{\pi}^{x} \cos(x+y)\,dy$

$I = \int_{\pi}^{2\pi} dx\,[\sin(x+y)]\Big|_{\pi}^{x} = \int_{\pi}^{2\pi} (\sin 2x - \sin(x+\pi))\,dx$

$= -\frac{\cos 2x}{2} + \cos(x+\pi)\Big|_{\pi}^{2\pi}$

$= \cos 3\pi - \cos 2\pi - \frac{1}{4}\underbrace{(\cos 4\pi - \cos 2\pi)}_{0} = -1-1 = -2$

116. $\iint_D \frac{y^3}{x^2+y^2}\,dx\,dy$ $\quad D:\ y=4,\ y=8,\ y=x,\ x=0$



$D: \begin{cases} 4 \le y \le 8 \\ 0 \le x \le y \end{cases}$

$\int \frac{dx}{y^2+x^2} = \frac{1}{y^2}\int \frac{dx}{1+\left(\frac{x}{y}\right)^2}$ $\quad \frac{x}{y}=t,\ dx = y\,dt$

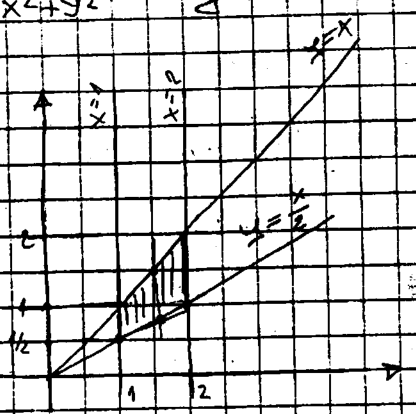
$I = \int_4^8 dy \int_0^y \frac{y^3}{x^2+y^2}\,dx$ $\quad = \frac{1}{y^2}\cdot y\int \frac{dt}{1+t^2} = \frac{1}{y}\operatorname{arctg}\frac{x}{y}$

$I = \int_4^8 dy\,\frac{y^3}{y}\operatorname{arctg}\frac{x}{y} = \int_4^8 y^2(\operatorname{arctg}1 - \operatorname{arctg}0)\,dy$

$= \int_4^8 y^2\left(\frac{\pi}{4}-0\right)dy = \frac{\pi}{4}\cdot\frac{y^3}{3}\Big|_4^8 = \frac{\pi}{12}(8^3-4^3) = \frac{\pi}{12}(4^3 2^3 - 4^3) = \frac{4^3(2^3-1)\pi}{12}$

$I = \frac{4^2\cdot\cancel{4}}{\cancel{12}\,3}\cdot 7\pi = \frac{112}{3}\pi$

17. $\iint_D \frac{x}{x^2+y^2}\,dx\,dy$ $x=1,\ x=2,\ y=x,\ x=2y$



$D:\begin{cases} 1 \le x \le 2 \\ \frac{x}{2} \le y \le x \end{cases}$

$$I = \int_1^2 dx \int_{\frac{x}{2}}^{x} \frac{x}{x^2+y^2}\,dy$$

$\operatorname{tg} x = \frac{1}{2}$

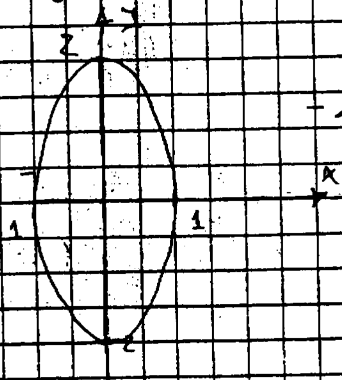
$\operatorname{arctg} 1 = \frac{\pi}{4}$ $\operatorname{arctg}\frac{1}{2} = x$

$$I = \int_1^2 dx \left[\frac{1}{x}\operatorname{arctg}\frac{y}{x}\right]\Big|_{x/2}^{x} \cdot x = \int_1^2 \left(\operatorname{arctg} 1 - \operatorname{arctg}\frac{1}{2}\right) dx$$

$$I = \left(\frac{\pi}{4} - \operatorname{arctg}\frac{1}{2}\right)$$

18. $\iint_D (xy)\,dx\,dy$ $D = \{(x,y) \mid 4x^2+y^2 \le 4\}$

$D:\ x^2 + \frac{y^2}{4} = 1$



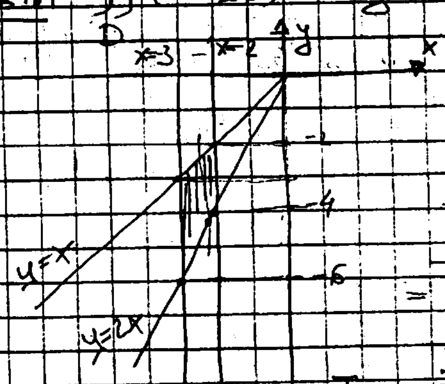
$-1 \le x \le 1$

$-2\sqrt{1-x^2} \le y \le 2\sqrt{1-x^2}$

$$I = \int_{-1}^{1} dx \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} xy\,dy$$

$$I = \int_{-1}^{1} dx\, \frac{xy^2}{2}\Big|_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} = \int_{-1}^{1} \frac{x}{2}\left[\cancel{4(1-x^2)} - \cancel{4(1-x^2)}\right] dx = 0$$

21. $\iint_D (x+2y)\,dx\,dy$ $x=-2,\ x=-3,\ y=x,\ y=2x$



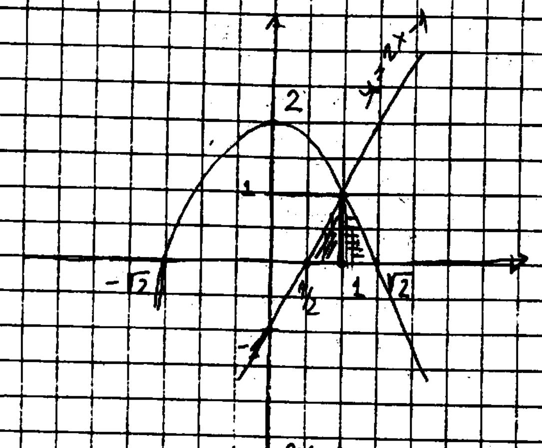
$D:\begin{cases} -3 \le x \le -2 \\ 2x \le y \le x \end{cases}$

$$I = \int_{-3}^{-2} dx \int_{2x}^{x} (x+2y)\,dy = \int_{-3}^{-2} dx\,(xy+y^2)\Big|_{2x}^{x}$$

$$= \int_{-3}^{-2} (\cancel{x^2} + \cancel{x^2} - \cancel{2x^2} - 4x^2)\,dx = -\frac{4}{3}x^3\Big|_{-3}^{-2} = -\frac{4}{3}(-8+27)$$

$$I = -\frac{76}{3}$$

$\boxed{22.}$ $\iint\limits_D (x-y)\,dx\,dy$ $\quad y=2-x^2,\ y=2x-1$



$2x-1=2-x^2$

$x^2+2x-3=0 \quad x_{1,2}=\frac{-2\pm4}{2} \quad x_1=-3,\ x_2=1$

$D_1:\begin{cases} 1/2 \le x \le 1 \\ 0 \le y \le 2x-1 \end{cases}$

$D_2:\begin{cases} 1 \le x \le \sqrt{2} \\ 0 \le y \le 2-x^2 \end{cases}$

$$I=\int_{1/2}^{1}dx\int_0^{2x-1}(x-y)\,dy+\int_1^{\sqrt{2}}dx\int_0^{2-x^2}(x-y)\,dy$$

$$=\int_{1/2}^{1}dx\left[xy-\frac{y^2}{2}\right]\Bigg|_0^{2x-1}+\int_1^{\sqrt{2}}dx\left(xy-\frac{y^2}{2}\right)\Bigg|_0^{2-x^2}$$

$$=\int_{1/2}^{1}dx\left[x(2x-1)-\frac{1}{2}(2x-1)^2\right]+\int_1^{\sqrt{2}}dx\left[x(2-x^2)-\frac{1}{2}(2-x^2)^2\right]$$

$$=\int_{1/2}^{1}dx\left(\cancel{2x^2}-x-\cancel{2x^2}+2x-\frac{1}{2}\right)+\int_1^{\sqrt{2}}dx\left(2x-x^3-2+2x^2-\frac{x^4}{2}\right)$$

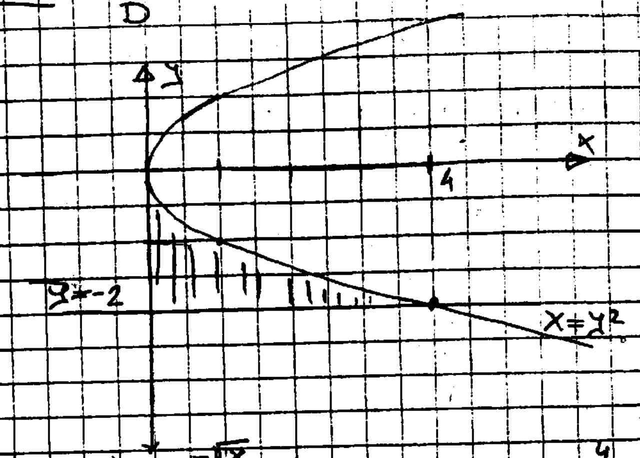
$$=\int_{1/2}^{1}\left(x-\frac{1}{2}\right)dx+\int_1^{\sqrt{2}}\left(\frac{1}{2}x^4-x^3+2x^2+2x\right)dx$$

$$=\frac{x^2}{2}+\frac{1}{2}x\Bigg|_{1/2}^{1}+\frac{x^5}{10}-\frac{x^4}{4}+\frac{2}{3}x^3+x^2\Bigg|_1^{\sqrt{2}}$$

$$=\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\frac{1}{8}-\frac{1}{4}\quad\frac{4\sqrt{2}}{10}-\cancel{1}+\frac{4\sqrt{2}}{3}+2=\frac{-1-2}{8}+\frac{16}{8}+\frac{40\sqrt{2}-12\sqrt{2}}{30}=\frac{13}{8}+\frac{28}{30}\sqrt{2}$$

$$I=\frac{13}{8}+\frac{14}{15}\sqrt{2}$$

23. $\iint\limits_D (3x^2-2xy+y)\,dx\,dy$ $\qquad x=0,\ x=y^2,\ y=-2$



$$D=\begin{cases}0\le x\le 4\\ -2\le y\le -\sqrt{x}\end{cases}$$

$$I=\int_0^4 dx\int_{-2}^{-\sqrt{x}}(3x^2-2xy+y)\,dy=\int_0^4 dx\left(3x^2y-xy^2+\frac{y^2}{2}\right)\Big|_{-2}^{-\sqrt{x}}$$

$$=\int_0^4\left[-3x^2\sqrt{x}-x^2+\frac{x}{2}-(-6x^2-4x+2)\right]dx$$

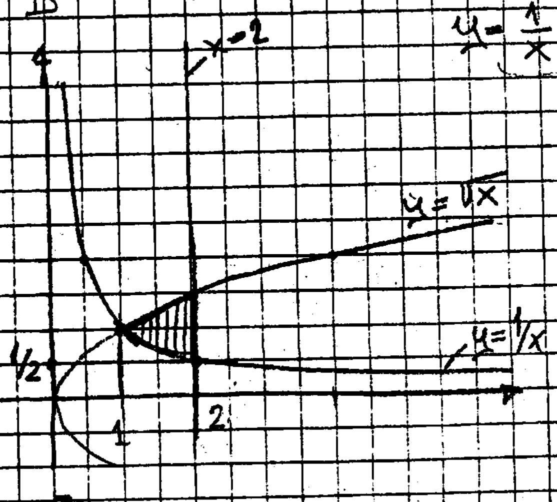
$$=\int_0^4\left(-3x^{5/2}-x^2+\frac{x}{2}+6x^2+4x-2\right)dx=\int_0^4\left(-3x^{5/2}+5x^2+\frac{9x}{2}-2\right)dx$$

$$=-\frac{6}{7}x^{7/2}+\frac{5}{3}x^3+\frac{9}{4}x^2-2x\Big|_0^4=-\frac{6}{7}\sqrt{4^7}+\frac{5}{3}4^3+\frac{9}{4}4^2-8$$

$$=-\frac{6}{7}2^7+\frac{5}{3}\cdot 4^3+28=-\frac{768}{7}+\frac{320}{3}+28=\frac{-64}{21}+28=25-\frac{1}{21}$$

24. $\iint\limits_D y\ln x\,dx\,dy$ $\qquad xy=1,\ y=\sqrt{x},\ x=2$

$$y=\frac{1}{x}\quad y^2=x,\quad x=2$$



$$D=\begin{cases}1\le x\le 2\\ \frac{1}{x}\le y\le \sqrt{x}\end{cases}$$

$$I=\int_1^2 dx\int_{1/x}^{\sqrt{x}} y\ln x\,dy$$

$$I=\int_1^2 \ln x\,dx\,\frac{y^2}{2}\Big|_{1/x}^{\sqrt{x}}$$

$$I=\int_1^2 \ln x\,dx\,\frac{1}{2}\left(x-\frac{1}{x^2}\right)\rightarrow I=\frac{1}{2}\underbrace{\int_1^2 x\ln x\,d}_{J_1}-\frac{1}{2}\underbrace{\int\frac{\ln x}{x^2}\,dx}_{J_2}$$

$J_1 = \int_1^2 x \ln x \, dx = \begin{matrix} u = \ln x & du = \frac{dx}{x} \\ dv = x\,dx & v = \frac{x^2}{2} \end{matrix}$

$J_1 = \frac{x^2 \ln x}{2} - \frac{1}{2}\int \frac{x^{\cancel{2}}\, dx}{\cancel{x}} = \frac{x^2 \ln x}{2}\Big|_1^2 - \frac{1}{4}x^2\Big|_1^2$

$J_1 = 2\ln 2 - \frac{3}{4}$

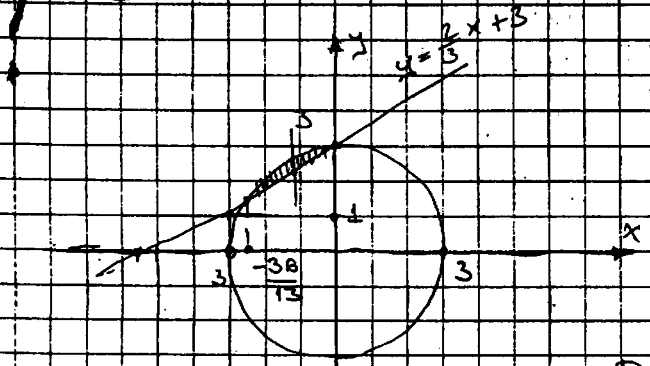
$J_2 = \int_1^2 \frac{\ln x}{x^2}\, dx = \left[\begin{matrix} u = \ln x & du = \frac{dx}{x} \\ dv = \frac{dx}{x^2} & v = -\frac{1}{x} \end{matrix}\right]$

$J_1 = -\frac{\ln x}{x} + \int_1^2 \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} = -\frac{1}{x}(\ln x + 1)$

$J_1 = -\left[\frac{1}{2}(\ln 2 + 1) - 1(\cancel{\ln 1} + 1)\right] = -\frac{1}{2}\ln 2 - \frac{1}{2} + 1$

$I = \ln 2 - \frac{3}{8} + \frac{1}{4}\ln 2 + \frac{1}{4} + \frac{1}{2} = \frac{5\ln 2}{4} + \frac{2+4-3}{8} = \frac{5\ln 2}{4} + \frac{5}{8} = \frac{10\ln 2 + 5}{8}$

1.25. $\iint_D (3x+y)\,dx\,dy \qquad D = \{(x,y) \mid x^2+y^2 \le 9 \wedge 3y \ge 2x+9\}$



$x^2 + \frac{4}{9}x^2 + 4x + \cancel{9} = \cancel{9}$

$\frac{13}{9}x^2 + 4x = 0$

$x\left(\frac{13}{9}x + 4\right) = 0$

$13x = -36$

$x = -\frac{36}{13}$

$D: \begin{cases} -\frac{36}{13} \le x \le 0 \\ \frac{2}{3}x + 3 \le y \le \sqrt{9-x^2} \end{cases}$

$I = \int_{-\frac{36}{13}}^{0} dx \int_{\frac{2x}{3}+3}^{\sqrt{9-x^2}} (3x+y)\,dy = \int_{-\frac{36}{13}}^{0} dx \left(3xy + \frac{y^2}{2}\right)\Bigg|_{\frac{2x}{3}+3}^{\sqrt{9-x^2}}$

$= \int_{-\frac{36}{13}}^{0} \left(3x\sqrt{9-x^2} + \frac{9-x^2}{2} - 2x^2 - 9x + \frac{4x^2+36x+81}{18}\right) dx$

$= \int_{-\frac{36}{13}}^{0} \left(3x\sqrt{9-x^2} + \frac{\cancel{81} - 9x^2 - 36x^2 - \cancel{162x} + 4x^2 + \cancel{36x} + \cancel{81}}{18}\right) dx$

$$\int_{-\frac{36}{13}}^{0}(3x\sqrt{9-x^2})\,dx + \int_{-\frac{36}{13}}^{0} -\frac{5}{18}x^2\,dx$$

$$J_1 = \int 3x\sqrt{9-x^2}\,dx = \Big| \quad 9-x^2 = t \quad -2x\,dx = dt \Rightarrow x\,dx = -\frac{dt}{2}$$

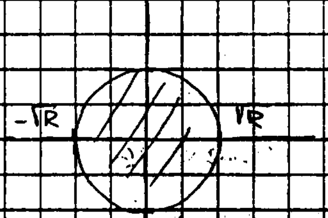
$$= -\frac{3}{2}\int t^{1/2}\,dt = -\frac{3}{2}\,\frac{t^{3/2}}{\frac{3}{2}} = -\sqrt{t^3} = -\sqrt{(9-x^2)^3}$$

$$J_1 = -\sqrt{(9-x^2)^3}\,\Big|_{-\frac{36}{13}}^{0} = -\Big[\sqrt{9^3} - \sqrt{\frac{117-1296}{169}}$$

26. $\iint_D y^2\sqrt{R^2-x^2}\,dx\,dy \qquad D = \{(x,y) \mid x^2+y^2 \le R\}$

?

$$D: \begin{cases} -\sqrt{R} \le x \le \sqrt{R} \\ -\sqrt{R-x^2} \le y \le \sqrt{R-x^2} \end{cases}$$



$$I = \int_{-\sqrt{R}}^{\sqrt{R}} dx \int_{-\sqrt{R-x^2}}^{\sqrt{R-x^2}} y^2\sqrt{R^2-x^2}\,dy = \int_{-\sqrt{R}}^{\sqrt{R}} \sqrt{R^2-x^2}\;\frac{y^3}{3}\Big|_{-\sqrt{R-x^2}}^{\sqrt{R-x^2}} dx$$

$$= \int_{-\sqrt{R}}^{\sqrt{R}} \sqrt{R^2-x^2}\cdot\frac{1}{3}\Big[(R-x^2)\sqrt{R-x^2} + (R-x^2)\sqrt{R-x^2}\Big]dx$$

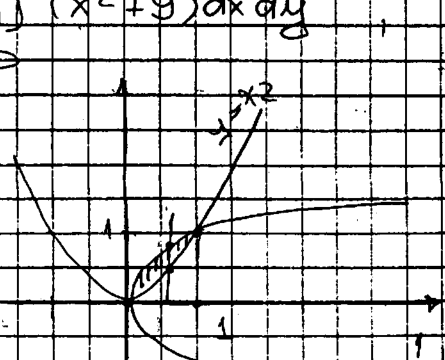
$$= \int_{-\sqrt{R}}^{\sqrt{R}} \frac{2}{3}(R-x^2)\sqrt{R-x^2}\sqrt{R^2-x^2} = \ldots = ?$$

27. $\iint_D (x^2+y)\,dx\,dy, \quad y = x^2, \quad x = y^2 \Rightarrow x^4 = x \quad x^4 - x = 0$

$x(x^3-1) = 0 \quad x_1 = 0$

$x_2 = 1$

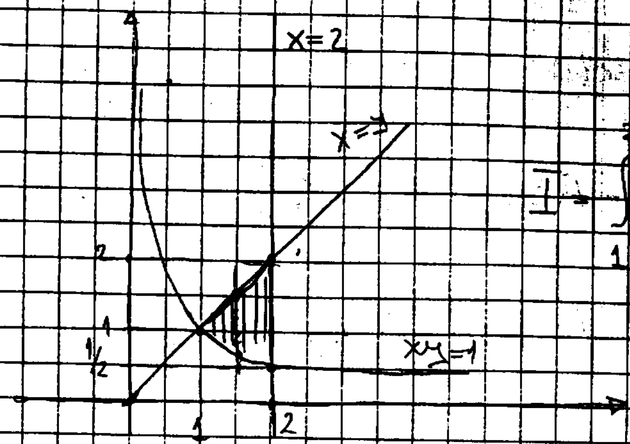
$$D: \begin{cases} 0 \le x \le 1 \\ x^2 \le y \le \sqrt{x} \end{cases}$$



$$I = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (x^2+y)\,dy = \int_0^1 dx\,\Big(x^2y + \frac{y^2}{2}\Big)\Big|_{x^2}^{\sqrt{x}}$$

$$I = \int_0^1\Big(x^2\sqrt{x} + \frac{x}{2} - x^4 - \frac{x^4}{2}\Big)dx = \int_0^1\Big(x^{5/2} - \frac{3}{2}x^4 + \frac{x}{2}\Big)dx = \frac{2}{7}x^{7/2} - \frac{3}{10}x^5 + \frac{x^2}{4} = \frac{40-42+35}{140} = \frac{33}{140}$$

29. $\iint_D \frac{x^2}{y^2}\,dx\,dy$ $\quad x=2 \quad x=y,\ xy=1$



$$D:\begin{cases} 1 \le x \le 2 \\ \frac{1}{x} \le y \le x \end{cases}$$

$$I = \int_1^2 dx \int_{1/x}^{x} \frac{x^2}{y^2}\,dy$$

$$I = \int_1^2 \left(-\frac{x^2}{y}\right)\Big|_{1/x}^{x} dx$$

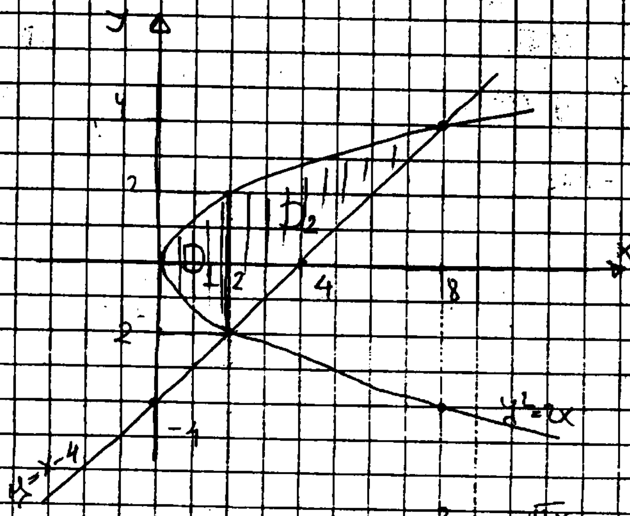
$$I = \int_1^2 -x^2\left(\frac{1}{x} - x\right)dx = \int_1^2 (x^3 - x)\,dx = \frac{x^4}{4} - \frac{x^2}{2}\Big|_1^2 = 4 - 2 - \frac{1}{4} + \frac{1}{2} = 2\frac{1}{4} = \frac{9}{4}$$

30. $\iint_D xy\,dx\,dy$ $\quad y = x-4 \quad y^2 = 2x$

$$x^2 - 8x + 16 - 2x = 0 \qquad x^2 - 10x + 16 = 0$$

$$x_{1/2} = \frac{10 \pm \sqrt{100-64}}{2} = \frac{10 \pm 6}{2}$$

$$x_1 = 8 \quad x_2 = 2$$



$$D_1:\begin{cases} 0 \le x \le 2 \\ -\sqrt{2x} \le y \le \sqrt{2x} \end{cases}$$

$$D_2:\begin{cases} 2 \le x \le 8 \\ x-4 \le y \le \sqrt{2x} \end{cases}$$

$$\iint_D = \iint_{D_1} + \iint_{D_2} = \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} xy\,dy + \int_2^8 dx \int_{x-4}^{\sqrt{2x}} xy\,dy$$

$$= \int_0^2 dx\, \frac{xy^2}{2}\Big|_{-\sqrt{2x}}^{\sqrt{2x}} + \int_2^8 dx\, \frac{xy^2}{2}\Big|_{x-4}^{\sqrt{2x}}$$

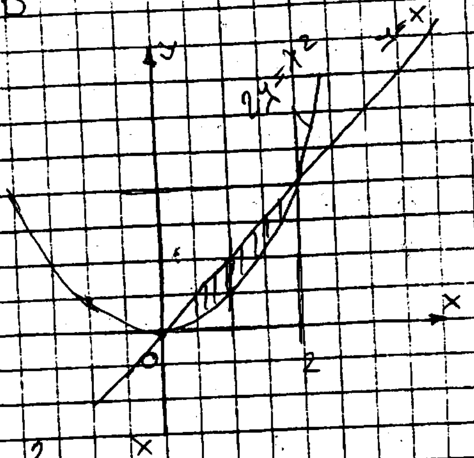
$$= \underbrace{\int_0^2 \frac{x}{2}(2x - 2x)\,dx}_{0} + \int_2^8 \frac{x}{2}(2x - x^2 + 8x - 16)\,dx$$

$$= \int_2^8 \left(-\frac{x^3}{2} + 5x^2 - 8x\right)dx = -\frac{x^4}{8} + \frac{5x^3}{8} - 4x^2\Big|_2^8 = \ldots$$

$\boxed{31.}$ $\iint\limits_D x\sqrt{x^2+y^2}\,dx\,dy$ $\quad y=x \quad x^2=2y \rightarrow \frac{x^2}{2}=x \quad x^2-2x=0$

$x(x-2)=0$

$x_1=0 \quad x_2=2$



$$D:\begin{cases} 0\le x\le 2 \\ \frac{x^2}{2}\le y\le x\end{cases}$$

$$I=\int_0^2 dx\cdot x\int_{\frac{x^2}{2}}^{x}\sqrt{x^2+y^2}\,dy$$

$M=\int\sqrt{x^2+y^2}\,dy \qquad y=x\,\mathrm{sh}\,t, \quad dy=x\,\mathrm{ch}\,t\,dt$

$=\int\sqrt{x^2+x^2\mathrm{sh}^2t}\cdot x\,\mathrm{ch}\,t\,dt$

$=\int\sqrt{x^2(1+\mathrm{sh}^2t)}\;x\,\mathrm{ch}\,t\,dt$

$=\int x^2\mathrm{ch}^2t\,dt=x^2\int\frac{1+\mathrm{ch}\,2t}{2}\,dt$

$=x^2\cdot\frac{1}{2}\left[t+\frac{\mathrm{sh}\,2t}{2}\right] \qquad c-s=1, \quad c=1+s$

$=\frac{x^2}{2}t+\frac{x^2}{4}\mathrm{sh}\,2t=\frac{x^2}{2}t+\frac{x^2}{2}\mathrm{sh}\,t\,\mathrm{ch}\,t$

$t=\mathrm{arsh}\,\frac{y}{x} \qquad \mathrm{sh}\,t=\frac{y}{x} \qquad \mathrm{ch}\,t=\frac{\sqrt{x^2+y^2}}{x}$

$t=\ln\left|\frac{y+\sqrt{y^2+x^2}}{x}\right| \qquad \mathrm{ch}\,t\cdot\mathrm{sh}\,t=\frac{y\sqrt{x^2+y^2}}{x^2}$

$$\boxed{M=\frac{x^2}{2}\ln\left|\frac{y+\sqrt{y^2+x^2}}{x}\right|+\frac{y}{2}\sqrt{x^2+y^2}}$$

$M(x)=\frac{x^2}{4}\ln\left|\frac{x+\sqrt{2x^2}}{x}\right|+\frac{x}{2}\sqrt{2x^2}$

$M(x)=\frac{x^2}{2}\ln|1+\sqrt{2}|+\frac{x^2}{2}\sqrt{2}$

$M(x)=\frac{x^2}{2}\ln|1+\sqrt{2}|+\frac{x^2}{\sqrt{2}}$

$M\left(\frac{x^2}{2}\right)=\frac{x^2}{2}\ln\left|\frac{\frac{x^2}{2}+\sqrt{\frac{x^4}{4}+x^2}}{x}\right|+\frac{x^2}{4}\sqrt{x^2+\frac{x^4}{4}}$

$=\frac{x^2}{2}\ln\left|\frac{\frac{x^2+\sqrt{x^4+4x^2}}{2}}{x}\right|+\frac{x^2}{4}\sqrt{\frac{4x^2+x^4}{4}}$

$=\frac{x^2}{2}\ln\left|\frac{x^2+x\sqrt{x^2+4}}{2x}\right|+\frac{x^2}{4}\cdot\frac{x\sqrt{4+x^2}}{2}$

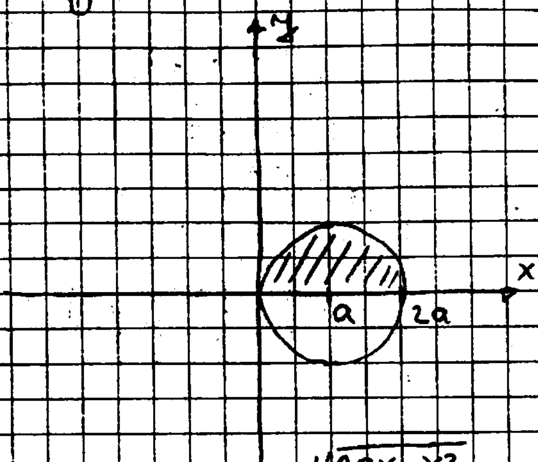
$=\frac{x^2}{2}\ln\left|\frac{x+\sqrt{x^2+4}}{2}\right|+\frac{x^3\sqrt{x^2+4}}{8}$

32. $\iint_D x^2 y\,dx\,dy$ $\qquad y=0 \qquad y=\sqrt{2ax-x^2}$

$y^2 = 2ax - x^2$

$x^2+y^2-2ax=0$

$(x-a)^2+y^2=a^2$

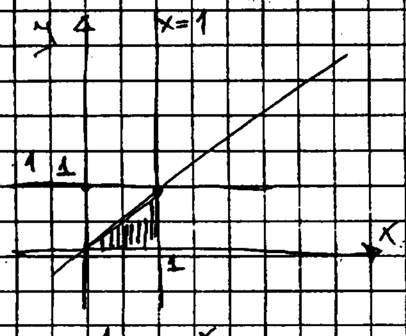


$$D:\begin{cases} 0 \le x \le 2a \\ 0 \le y \le \sqrt{2ax-x^2} \end{cases}$$

$$I=\int_0^{2a} x^2\,dx \int_0^{\sqrt{2ax-x^2}} y\,dy = \int_0^{2a} x^2\cdot\frac{y^2}{2}\Big|_0^{\sqrt{2ax-x^2}} dx = \int_0^{2a} \frac{x^2(2ax-x^2)}{2}\,dx$$

$$=\int_0^{2a}\frac{2ax^3-x^4}{2}\,dx = \frac{ax^4}{4}-\frac{x^5}{10}\Big|_0^{2a} = \frac{16a^5}{4}-\frac{32a^5}{10} = 4a^5-\frac{32a^5}{10}=\frac{4}{5}a^5$$

33. $\iint_D \sqrt{4x^2-y^2}\,dx\,dy$ $\qquad y=0,\ x=1,\ y=x$



$$D:\begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$$

$$I=\int_0^1 dx\int_0^x \sqrt{4x^2-y^2}\,dy$$

$M=\int \sqrt{4x^2-y^2}\,dy$ $\qquad \sin t=\frac{y}{2x} \quad \cos t=\frac{\sqrt{4x^2-y^2}}{2x}$

$y=2x\sin t$

$dy=2x\cos t\,dt$

$=\int\sqrt{4x^2-4x^2\sin^2 t}\;2x\cos t\,dt$

$=\int 4x^2\cos^2 t\,dt = 4x^2\int\frac{1+\cos 2t}{2}\,dt$

$=2x^2 t + x^2\sin 2t$

$=2x^2\arcsin\frac{y}{2x}+2x^2\sin t\cos t$

$=2x^2\arcsin\frac{y}{2x}+\cancel{2x^2}\cdot\frac{y}{\cancel{2x}}\cdot\frac{\sqrt{4x^2-y^2}}{2\cancel{x}}$

$M=2x^2\arcsin\frac{y}{2x}+\frac{y}{2}\sqrt{4x^2-y^2}$

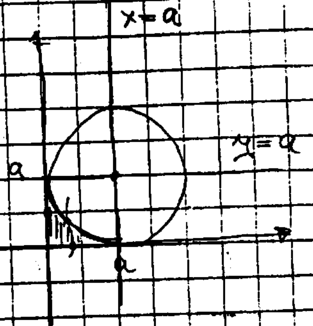
$M(x)=2x^2\arcsin\frac{1}{2}+\frac{x}{2}\sqrt{3}\,x$

$M(x)=\cancel{2}x^2\frac{\pi}{\cancel{6}_3}+\frac{\sqrt{3}}{2}x^2$

$M(0)=2x^2\arcsin\overset{0}{\cancel{0}}+\frac{\overset{0}{\cancel{y}}}{\cancel{2}}\cancel{2x}$

$$I = \int_0^1 \left(\frac{\pi}{3}x^2 + \frac{\sqrt{3}}{2}x^2\right)dx = \frac{\pi}{3}\cdot\frac{x^3}{3} + \frac{\sqrt{3}}{2}\cdot\frac{x^3}{3} = \left(\frac{\pi}{9}+\frac{\sqrt{3}}{6}\right)x^3 = \frac{\pi}{9}+\frac{\sqrt{3}}{6}$$

35. $\displaystyle\iint_D \frac{dx\,dy}{\sqrt{2a-x}}$ $\quad D = \left\{ (x,y) \;\middle|\; \begin{array}{l} (x-a)^2+(y-a)^2 \geq a^2 \\ 0 \leq x \leq a \quad 0 \leq y \leq a \end{array} \right.$



$$(y-a)^2 = a^2-(x-a)^2$$

$$y-a = \pm\sqrt{\cancel{a^2}-x^2+2ax-\cancel{a^2}}$$

$$y = a \pm \sqrt{2ax-x^2}$$

$$D = \begin{cases} 0 \leq x \leq a \\ 0 \leq y \leq a-\sqrt{2ax-x^2} \end{cases}$$

$$I = \int_0^a dx \int_0^{a-\sqrt{2ax-x^2}} \frac{dy}{\sqrt{2a-x}} = \int_0^a \frac{dx}{\sqrt{2a-x}}\cdot y\Big|_0^{a-\sqrt{2ax-x^2}}$$

$$= \int_0^a \frac{a-\sqrt{x(2a-x)}}{\sqrt{2a-x}}\,dx = \underbrace{\int_0^a \frac{a\,dx}{\sqrt{2a-x}}}_{J_1} - \underbrace{\int_0^a \frac{\sqrt{x}\,\sqrt{2a-x}}{\sqrt{2a-x}}\,dx}_{J_2}$$

$$J_1 = \int_0^a \frac{a\,dx}{\sqrt{2a-x}} = \begin{array}{l} 2a-x=t \\ -dx=dt \end{array}$$

$$= \int \frac{-a\,dt}{t^{1/2}} = -a\int t^{-1/2}dt = -a(2\sqrt{t}) = -2a\sqrt{2a-x} = 2a(\sqrt{a}-\sqrt{2a})$$

$$J_2 = \int_0^a \sqrt{x}\,dx = \int_0^a x^{1/2}dx = \frac{2x^{3/2}}{3}\Big|_0^a = \frac{2}{3}\sqrt{a^3} = \frac{2a\sqrt{a}}{3}$$

$$I = 2a\sqrt{a}-2a\sqrt{2a}-\frac{2a\sqrt{a}}{3} = \frac{4a\sqrt{a}}{3}-2a\sqrt{2a}$$

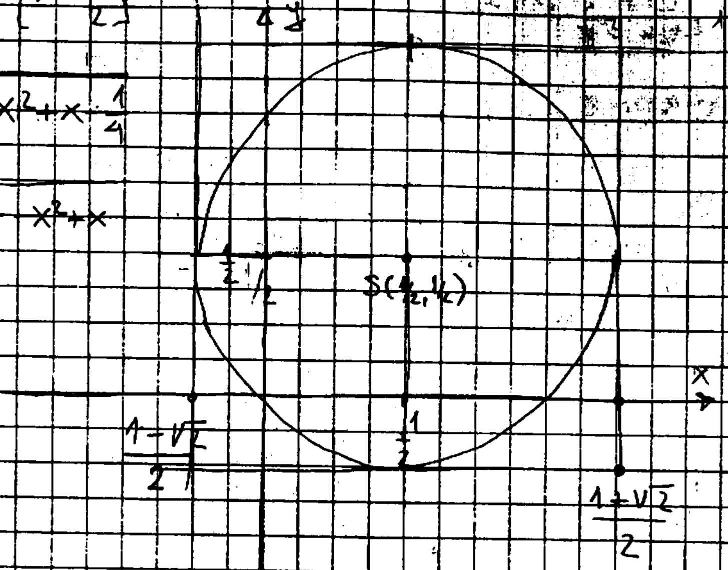
36. $\iint_D (x+y)\,dx\,dy \qquad D=\{(x,y) \mid x^2+y^2-x-y\le 0\}$

$$\left(x-\frac12\right)^2+\left(y-\frac12\right)^2=\frac12$$

$$\left[y-\frac12\right]^2=\frac12-\left[x-\frac12\right]^2$$

$$y-\frac12=\sqrt{\frac12-x^2+x-\frac14}$$

$$y=\frac12\pm\sqrt{\frac14-x^2+x}$$



$$\frac12+\frac1{\sqrt2}=\frac{1+\sqrt2}{2}$$

$$I=\int_{\frac{1-\sqrt2}{2}}^{\frac{1+\sqrt2}{2}}dx\int_{\frac12-\sqrt{\frac14-x^2+x}}^{\frac12+\sqrt{\frac14-x^2+x}}(x+y)\,dy=\int_{\frac{1-\sqrt2}{2}}^{\frac{1+\sqrt2}{2}}dx\left(xy+\frac{y^2}{2}\right)$$

$$x\left(\cancel{\frac12}+\sqrt{\frac14-x^2+x}-\cancel{\frac12}+\sqrt{\frac14-x^2+x}\right)+\frac12\left[\cancel{\frac14}+\sqrt{\frac14-x^2+x}+\cancel{\frac14}-\cancel{x^2}+\cancel{x}\right.$$
$$\left.-\cancel{\frac14}+\sqrt{\frac14-x^2+x}-\cancel{\frac14}+\cancel{x^2}-\cancel{x}\right]$$

$$2x\sqrt{\frac14-x^2+x}+\sqrt{\frac14-x^2+x}$$

$$=(2x+1)\sqrt{\frac14-x^2+x}$$

$$I=\int_{\frac{1-\sqrt2}{2}}^{\frac{1+\sqrt2}{2}}(2x+1)\sqrt{\left(x^2+x+\frac14\right)}\,dx \qquad ?$$

139. $\iint\limits_D (x+y)\,dx\,dy$, $D=\{(x,y) \mid 0\le x\le 2,\ 0\le y\le 2\}$

$$I=\int_0^2 dx\int_0^2 (x+y)\,dy=\int_0^2 dx\left(xy+\frac{y^2}{2}\right)\Big|_0^2=\int_0^2 dx\,(2x+2)$$

$$=x^2+2x\Big|_0^2=4+4=8$$

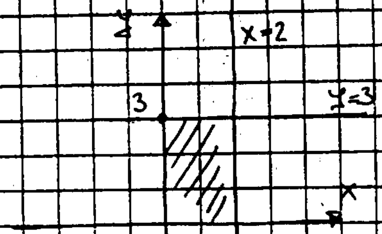
6.3. Промена поретка интеграције у двојном интегралу.

А) Мењајући поредак интеграције израчунати следеће двојне инт.

1) $\int_0^2 dx\int_0^3 (x^2+2xy)\,dy$

$$D:\begin{cases}0\le x\le 2\\ 0\le y\le 3\end{cases}$$

$$I=\iint\limits_D f(x,y)\,dx\,dy=\int_0^3 dy\int_0^2 (x^2+2xy)\,dx$$

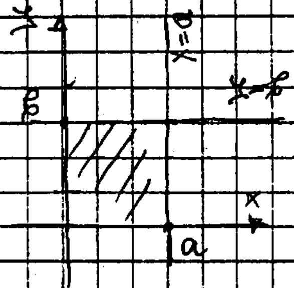


$$=\int_0^3 dy\left(\frac{x^3}{3}+x^2y\right)\Big|_0^2$$

$$=\int_0^3\left(\frac{8}{3}+4y\right)dy=\frac{8}{3}y+2y^2\Big|_0^3$$

$$I=\frac{8}{3}\cdot 3+2\cdot 3^2=8+18=26$$

2) $\int_0^a dx\int_0^b xy\,dy$



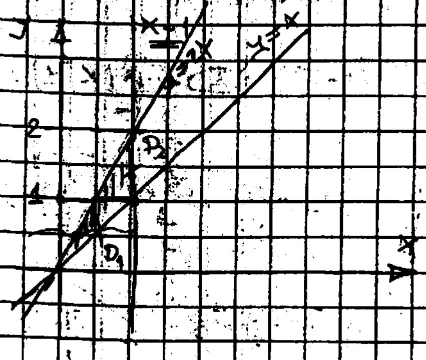
$$D^*=\begin{cases}0\le y\le b\\ 0\le x\le a\end{cases}$$

$$D\begin{cases}0\le x\le a\\ 0\le y\le b\end{cases}$$

$$I=\int_0^a dx\int_0^b xy\,dy=\iint\limits_{D^*} f(x,y)\,dx\,dy=\int_0^b dy\int_0^a xy\,dx$$

$$=\int_0^b dy\left(\frac{x^2}{2}y\Big|_0^a\right)=\int_0^b\frac{a^2y}{2}\,dy=\frac{a^2y^2}{4}\Big|_0^b=\frac{a^2b^2}{4}$$

3. $\int_0^1 dx \int_x^{2x} (x-y+1)\,dy$



$$D = \begin{cases} 0 \le x \le 1 \\ x \le y \le 2x \end{cases}$$

$$D_1^* = \begin{cases} 0 \le y \le 1 \\ \frac{y}{2} \le x \le y \end{cases} \qquad D_2^* = \begin{cases} 1 \le y \le 2 \\ \frac{y}{2} \le x \le 1 \end{cases}$$

$$I = \iint_{D^*} = \iint_{D_1^*} + \iint_{D_2^*} = \int_0^1 dy \int_{y/2}^{y} (x-y+1)\,dx + \int_1^2 dy \int_{y/2}^{1} (x-y+1)\,dx$$

$$I = \int_0^1 \left(\frac{x^2}{2} - yx + x\right)\Big|_{y/2}^{y} dy + \int_1^2 \left(\frac{x^2}{2} - yx + x\right)\Big|_{y/2}^{1} dy$$

$$I = \int_0^1 \left[\cancel{\frac{y^2}{2}} - \cancel{y^2} + y - \frac{y^2}{8} + \cancel{\frac{y^2}{2}} - \frac{y}{2}\right] dy + \int_1^2 \left[\frac{1}{2} - y + 1 - \frac{y^2}{8} + \frac{y^2}{2} - \frac{y}{2}\right] dy$$

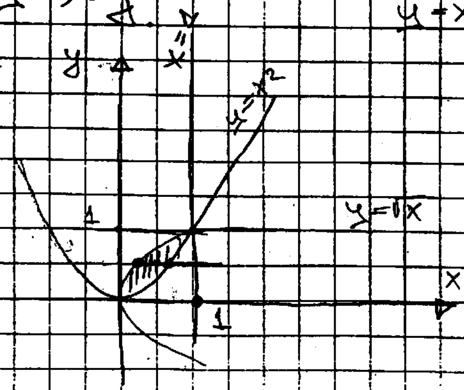
$$I = \int_0^1 \left(\frac{y}{2} - \frac{y^2}{8}\right) dy + \int_1^2 \left(\frac{3y^2}{8} - \frac{3}{2}y + \frac{3}{2}\right) dy$$

$$= \frac{y^2}{4} - \frac{y^3}{24}\Big|_0^1 + \frac{3y^3}{24} - \frac{3}{4}y^2 + \frac{3}{2}y\Big|_1^2 = \frac{1}{4} - \frac{1}{24} + 1 - \cancel{3} + \cancel{3} - \frac{1}{8} + \frac{3}{4} - \frac{3}{2}$$

$$= 2 - \frac{1}{24} - \frac{1}{8} - \frac{3}{2} = \frac{48-1-3-36}{24} = \frac{8}{24} = \frac{1}{3}$$

4. $I = \int_0^1 dx \int_{x^2}^{\sqrt{x}} x^2(y-x)\,dy$ $y = x^2$ $y^2 = x$

$$D = \begin{cases} 0 \le x \le 1 \\ x^2 \le y \le \sqrt{x} \end{cases}$$



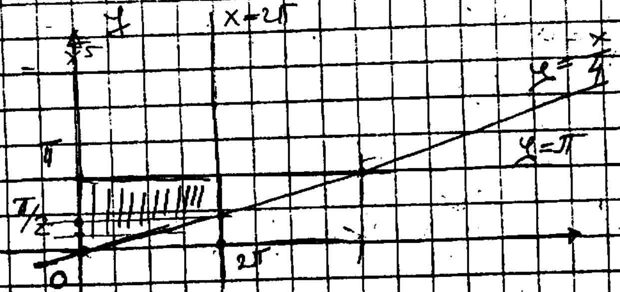
$$D^* = \begin{cases} 0 \le y \le 1 \\ y^2 \le x \le \sqrt{y} \end{cases}$$

$$I=\iint_{D^*} f(x,y)\,dx\,dy = \int_0^1 dy \int_{y^2}^{\sqrt{y}} (x^2y - x^3)\,dx = \int_0^1 dy \left(\frac{yx^3}{3} - \frac{x^4}{4}\right)\Bigg|_{y^2}^{\sqrt{y}}$$

$$= \int_0^1 \left[\frac{y^2\sqrt{y}}{3} - \frac{y^2}{4} - \frac{y^7}{3} + \frac{y^8}{4}\right] dy =$$

$$= \frac{2y^{7/2}}{21} - \frac{y^3}{12} - \frac{y^8}{24} + \frac{y^9}{36}\Bigg|_0^1 = \frac{2}{21} - \frac{1}{12} - \frac{1}{24} + \frac{1}{36} = \ldots$$

5. $\displaystyle\int_0^{2\pi} \cos^2 x\,dx \int_{\frac{x}{4}}^{\pi} y\,dy$



$$D=\begin{cases} 0 \le x \le 2\pi \\ \frac{x}{4} \le y \le \pi \end{cases}$$

$$D_1^*=\begin{cases} 0 \le y \le \frac{\pi}{2} \\ 0 \le x \le 4y \end{cases} \qquad D_2^*=\begin{cases} \frac{\pi}{2} \le y \le \pi \\ 0 \le x \le 2\pi \end{cases}$$

$$I=\iint_{D^*} f(x,y)\,dx\,dy = \iint_{D_1^*} f(x,y)\,dx\,dy + \iint_{D_2^*} f(x,y)\,dx\,dy$$

$$I=\int_0^{\pi/2} dy \int_0^{4y} \cos^2 x\, y\,dx + \int_{\pi/2}^{\pi} dy \int_0^{2\pi} \cos^2 x\, y\,dx$$

$$I=\int_0^{\pi/2} y\,dy \int_0^{4y} \frac{\cos 2x + 1}{2}\,dx + \int_{\pi/2}^{\pi} y\,dy \int_0^{2\pi} \frac{\cos 2x + 1}{2}\,dx$$

$$I=\int_0^{\pi/2} y\,dy \left[\frac{1}{2}\left(\frac{\sin 2x}{2} + x\right)\Bigg|_0^{4y}\right] + \int_{\pi/2}^{\pi} y\,dy \left[\frac{1}{2}\left(\frac{\sin 2x}{2} + x\right)\Bigg|_0^{2\pi}\right]$$

$$I=\int_0^{\pi/2} y\,dy\,\frac{1}{2}\left(\frac{\sin 8y}{2}\right) + \int_{\pi/2}^{\pi} y\,dy\,\frac{1}{2}(2\pi) = \int_0^{\pi/2} \frac{1}{4} y \sin 8y\,dy + \int_{\pi/2}^{\pi} \pi y\,dy$$

$u = y \quad du = dy$

$dv = \int \sin 8y\,dy$

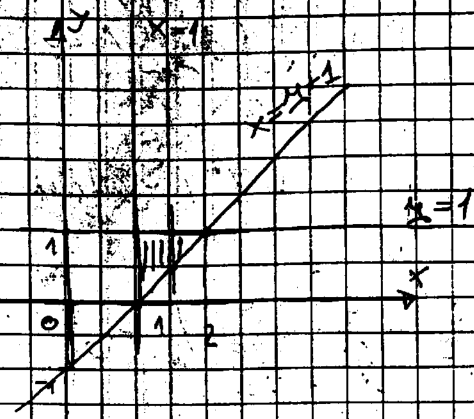
$v = \frac{-\cos 8y}{8}$

$$= \frac{1}{4}\left[\frac{y\cos 8y}{8}\Bigg|_0^{\pi/2} + \frac{1}{8}\int_0^{\pi/2} \cos 8y\,dy\right] + \pi\left(\pi^2 - \frac{\pi^2}{4}\right)$$

$$= -\frac{1}{32}\left[\frac{\pi}{2}\cos 4\pi\right] + \frac{1}{64}\sin 8y\Bigg|_0^{\pi/2} + \pi^3 - \frac{\pi^3}{1} = -\frac{\pi}{64} - \frac{\pi^3}{4} + \pi^3$$

6) $\int_0^1 dy \int_1^{y+1} \frac{\ln x}{x}\,dx$

$$D = \begin{cases} 0 \le y \le 1 \\ 1 \le x \le y+1 \end{cases}$$



$x - y = 1$

$$D^* = \begin{cases} 1 \le x \le 2 \\ x-1 \le y \le 1 \end{cases}$$

$$I = \int_1^2 dx \int_{x-1}^1 \frac{\ln x}{x}\,dy = \int_1^2 dx\,\frac{\ln x}{x}\left[y\Big|_{x-1}^1\right] = \int_1^2 dx\,\frac{\ln x}{x}(1-x+1)$$

$$= \int_1^2 \frac{(2-x)\ln x}{x}\,dx = \int_1^2 2\frac{\ln x}{x}\,dx - \int_1^2 \ln x\,dx$$

$$J_1 = \int_1^2 2\frac{\ln x}{x}\,dx = \left[\; t = \ln x \quad dt = \frac{dx}{x} \right]$$

$$= \int_1^2 2t\,dt = \ln^2 x\Big|_1^2 = \ln^2 2$$

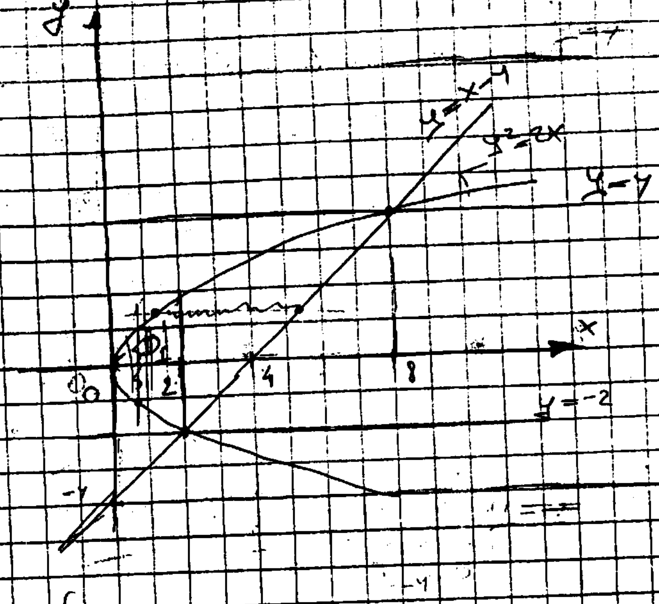
$$J_2 = \int_1^2 \ln x\,dx = \left[\begin{matrix} u = \ln x & du = \frac{dx}{x} \\ dv = dx & v = x \end{matrix}\right]$$

$$= x\ln x\Big|_1^2 - \int_1^2 dx = 2\ln 2 - x\Big|_1^2 = 2\ln 2 - 2 + 1 = 2\ln 2 - 1$$

$$I = \ln^2 2 - 2\ln 2 + 1$$

7. $\int_{-2}^{4} dy \int_{\frac{y^2}{2}}^{y+4} xy\,dx$

$$D = \begin{cases} -2 \le y \le 4 \\ \frac{y^2}{2} \le x \le y+4 \end{cases}$$



$y^2 = 2x \qquad y = x-4$

$x^2 - 8x + 16 - 2x = 0$

$x^2 - 10x + 16 = 0$

$x_{1/2} = \frac{10 \pm \sqrt{100-64}}{2} = \frac{10 \pm 6}{2}$

$x_1 = 8 \qquad x_2 = 2$

$y_1 = 4 \qquad y_2 = -2$

$$D_1^* = \begin{cases} 0 \le x \le 2 \\ -\sqrt{2x} \le y \le \sqrt{2x} \end{cases}$$

$$D_2^* = \begin{cases} 2 \le x \le 8 \\ x-4 \le y \le \sqrt{2x} \end{cases}$$

$$I = \int_0^2 dx \int_{-\sqrt{2x}}^{\sqrt{2x}} xy\,dy + \int_2^8 dx \int_{x-4}^{\sqrt{2x}} xy\,dy$$

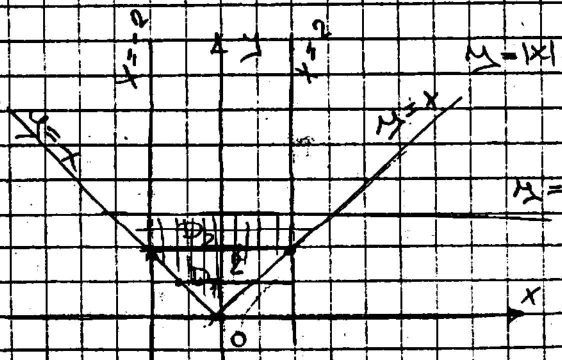
$$= \int_0^2 dx \left[\frac{xy^2}{2} \Big|_{-\sqrt{2x}}^{\sqrt{2x}} \right] + \int_2^8 dx \left[\frac{xy^2}{2} \Big|_{x-4}^{\sqrt{2x}} \right]$$

$$= \int_0^2 \frac{x}{2} dx \left[\cancel{4x^2 - 4x^2} \right] + \int_2^8 \frac{x}{2} dx \left[2x - x^2 + 8x - 16 \right]$$

$$= \int_2^8 \left(x^2 - \frac{x^3}{2} + 4x^2 - 8x \right) dx = \int_2^8 \left(-\frac{x^3}{2} + 5x^2 - 8x \right) dx = -\frac{x^4}{8} + \frac{5x^3}{3} - 4x^2 \Big|_2^8$$

8 $\int_{-2}^{2} dx \int_{|x|}^{3} (xy - y^2)\,dy$

$$y = |x| = \begin{cases} x > 0, & y = x \\ x < 0, & y = -x \end{cases}$$



$$D = \begin{cases} -2 \le x \le 2 \\ |x| \le y \le 3 \end{cases}$$

$$D_1^* = \begin{cases} 0 \le y \le 2 \\ -y \le x \le y \end{cases}$$

$$D_2^* = \begin{cases} 2 \le y \le 3 \\ -2 \le x \le 2 \end{cases}$$

$$I = \int_0^2 dy \int_{-y}^{y} (xy - y^2)\,dx + \int_2^3 dy \int_{-2}^{2} (xy - y^2)\,dx$$

$$= \int_0^2 dy \left[\left(yx^2 - xy^2 \right)\Big|_{-y}^{y} \right] + \int_2^3 dy \left[x^2 y - x y^2 \Big|_{-2}^{2} \right]$$

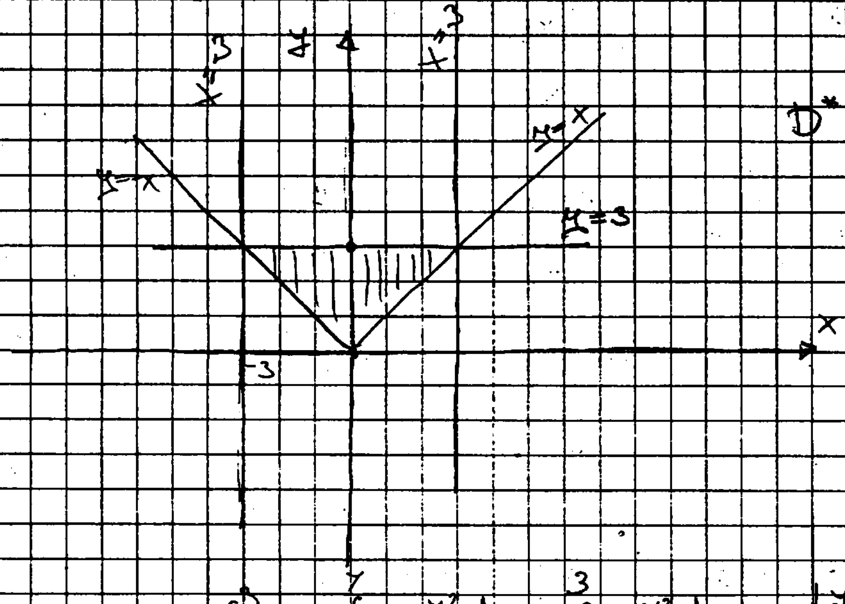
$$= \int_0^2 dy \left(\cancel{y^3} - y^3 - (y^3 + y^3) \right] + \int_2^3 dy \left[4y - 2y^2 - (4y + 2y^2) \right]$$

$$= \int_0^2 -2y^3\,dy + \int_2^3 -4y^2\,dy = -\frac{y^4}{2}\Big|_0^2 + \frac{4}{3} y^3 \Big|_2^3 = -8 - \frac{4}{3}(27 - 8)$$

$$= -8 - 36 + \frac{32}{3} = -44 + \frac{32}{3} = \frac{-132 + 32}{3} = -\frac{100}{3}$$

$\boxed{9.}$ $\int_{-3}^{0} dx \int_{-x}^{3} e^{-y^2} dy + \int_0^3 dx \int_x^3 e^{-y^2} dy$

$$D_1 = \begin{cases} -3 \le x \le 0 \\ -x \le y \le 3 \end{cases} \qquad D_2 = \begin{cases} 0 \le x \le 3 \\ x \le y \le 3 \end{cases}$$

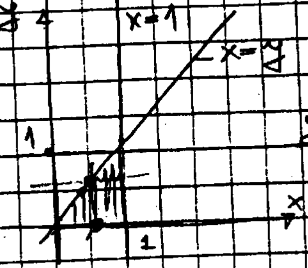


$$D^* = \begin{cases} 0 \le y \le 3 \\ -y \le x \le y \end{cases}$$

$$I = \int_0^3 dy \int_{-y}^{y} e^{-y^2} dx = \int_0^3 e^{-y^2} dy \cdot x \Big|_{-y}^{y} = \int_0^3 2y e^{-y^2} dy$$

$$= \left[t = -y^2 \quad dt = -2y\,dy \right] = -\int_0^{-9} e^t dt = -e^t = -(e^{-9} - e^0) = 1 - e^{-9}$$

10) $\int_0^1 dy \int_y^1 \frac{\sin x}{x} dx$



$D = \begin{cases} 0 \le y \le 1 \\ y \le x \le 1 \end{cases}$ $D^* = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le x \end{cases}$

$I = \int_0^1 dx \int_0^x \frac{\sin x}{x} dx$

Б) Заменити поредак интеграције у следећим двојним интегралима

1) $I = \int_1^3 dx \int_{-1}^4 f(x,y) dy$

$I = \iint_D f(x,y) = \int_{-1}^4 dy \int_1^3 f(x,y) dx$

2) $I = \int_{-2}^1 dx \int_0^3 f(x,y) dy$

$I = \iint_D f(x,y) = \int_0^3 dy \int_2^1 f(x,y) dx$

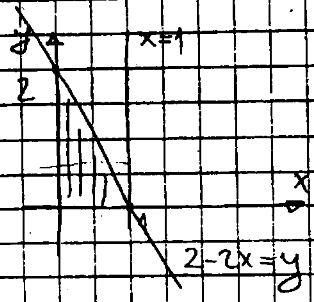
3) $I = \int_{-1}^3 dx \int_{-2}^2 f(x,y) dy$

$I = \iint_D f(x,y) = \int_{-2}^2 dy \int_1^3 f(x,y) dx$

4) $I = \int_2^3 dx \int_{-2}^{-1} f(x,y) dy$

$I = \iint_D f(x,y) = \int_{-2}^{-1} dy \int_2^3 f(x,y) dx$

5) $I = \int_0^1 dx \int_0^{2-2x} f(x,y) dy$



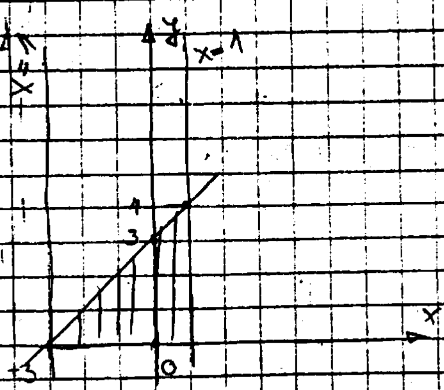
$D = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 2-2x \end{cases}$ $D^* = \begin{cases} 0 \le y \le 2 \\ 0 \le x \le 1-\frac{y}{2} \end{cases}$

$I = \iint_{D^*} f(x,y) dx dy = \int_0^2 dy \int_0^{1-\frac{y}{2}} f(x,y) dx$

6) $I = \int_{-3}^1 dx \int_0^{x+3} f(x,y) dy$

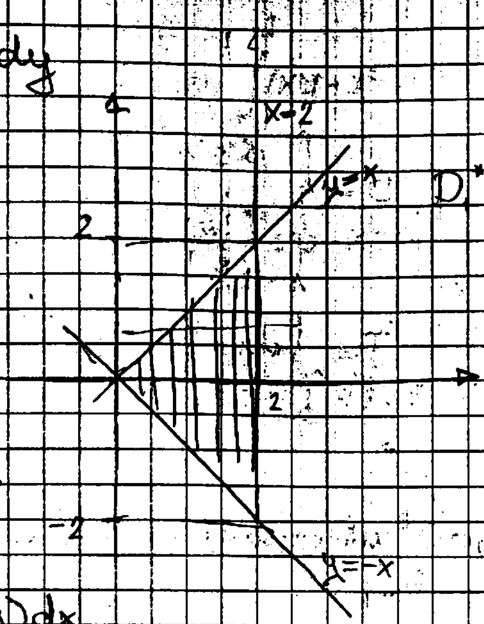
$D = \begin{cases} -3 \le x \le 1 \\ 0 \le y \le x+3 \end{cases}$ $I = \iint_{D^*} f(x,y) dx dy$

$D^* = \begin{cases} 0 \le y \le 4 \\ y-3 \le x \le 1 \end{cases}$ $= \int_0^4 dy \int_{y-3}^1 f(x,y) dx$



7. $I = \int_0^2 dx \int_{-x}^{x} f(x,y)\,dy$

$D = \begin{cases} 0 \le x \le 2 \\ -x \le y \le x \end{cases}$

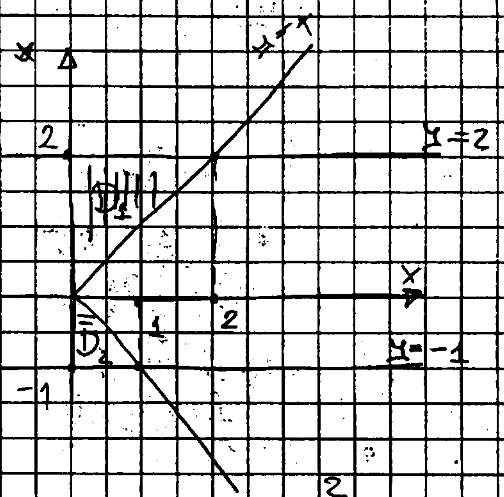


$D^* = \begin{cases} -2 \le y \le 0 \\ -y \le x \le 2 \end{cases}$

$D_2^* = \begin{cases} 0 \le y \le 2 \\ y \le x \le 2 \end{cases}$

$I = \iint_{D_1^*} + \iint_{D_2^*} = \int_{-2}^{0} dy \int_{-y}^{2} f(x,y)\,dx + \int_0^2 dy \int_y^2 f(x,y)\,dx$

8. $I = \int_{-1}^{2} dy \int_0^{|y|} f(x,y)\,dx$



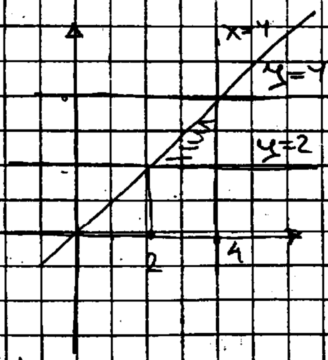
$D = \begin{cases} -1 \le y \le 2 \\ 0 \le x \le |y| \end{cases}$

$D_2^* = \begin{cases} 0 \le x \le 1 \\ -1 \le y \le -x \end{cases}$

$D_1^* = \begin{cases} 0 \le x \le 2 \\ x \le y \le 2 \end{cases}$

$I = \int_0^1 dx \int_{-1}^{-x} f(x,y)\,dy + \int_0^2 dx \int_x^2 f(x,y)\,dy$

9. $I = \int_2^4 dy \int_y^4 f(x,y)\,dx$

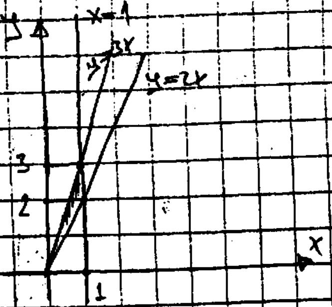


$D = \begin{cases} 2 \le y \le 4 \\ y \le x \le 4 \end{cases}$

$D^* = \begin{cases} 2 \le x \le 4 \\ 2 \le y \le x \end{cases}$

$I = \int_2^4 dx \int_2^x f(x,y)\,dy$

10. $\int_0^1 dx \int_{2x}^{3x} f(x,y)\,dy$



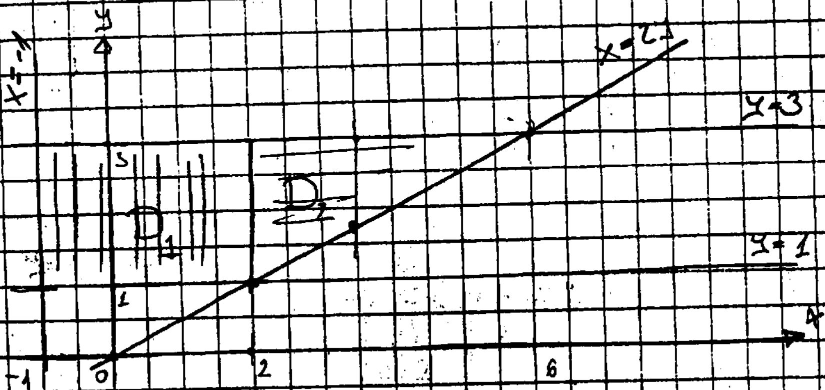
$D_1^* = \begin{cases} 0 \le y \le 2 \\ \frac{y}{3} \le x \le y/2 \end{cases}$

$D: \begin{cases} 0 \le x \le 1 \\ 2x \le y \le 3x \end{cases}$

$D_2^* = \begin{cases} 2 \le y \le 3 \\ \frac{y}{3} < x \le 1 \end{cases}$

$I = \int_0^2 dy \int_{y/3}^{y/2} f(x,y)\,dx + \int_2^3 dy \int_{y/3}^{1} f(x,y)\,dx$

12. $\int_1^3 dy \int_{-1}^{2y} f(x,y)\,dx$

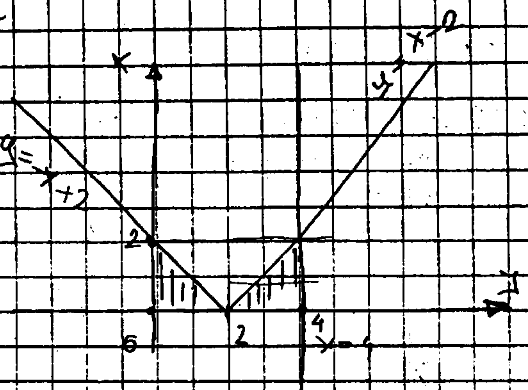


$D: \begin{cases} 1 \le y \le 3 \\ -1 \le x \le 2y \end{cases}$

$D_1^* = \begin{cases} -1 \le x \le 2 \\ 1 \le y \le 3 \end{cases}$ $D_2^* = \begin{cases} 2 \le x \le 6 \\ \frac{x}{2} \le y \le 3 \end{cases}$

$I = \int_{-1}^{2} dx \int_1^3 f(x,y)\,dy + \int_2^6 dx \int_{\frac{x}{2}}^{3} f(x,y)\,dy$

13. $\int_0^4 dx \int_0^{|x-2|} f(x,y)\,dy$



$D: \begin{cases} 0 \le x \le 4 \\ 0 \le y \le |x-2| \end{cases}$

$D_1^* = \begin{cases} 0 \le y \le 2 \\ 0 \le x \le 2-y \end{cases}$

$I = \int_0^2 dy \int_0^{2-y} f(x,y)\,dx + \int_0^2 dy \int_{y+2}^{4} f(x,y)\,dx$

$D_2^* = \begin{cases} 0 \le y \le 2 \\ y+2 \le x \le 4 \end{cases}$

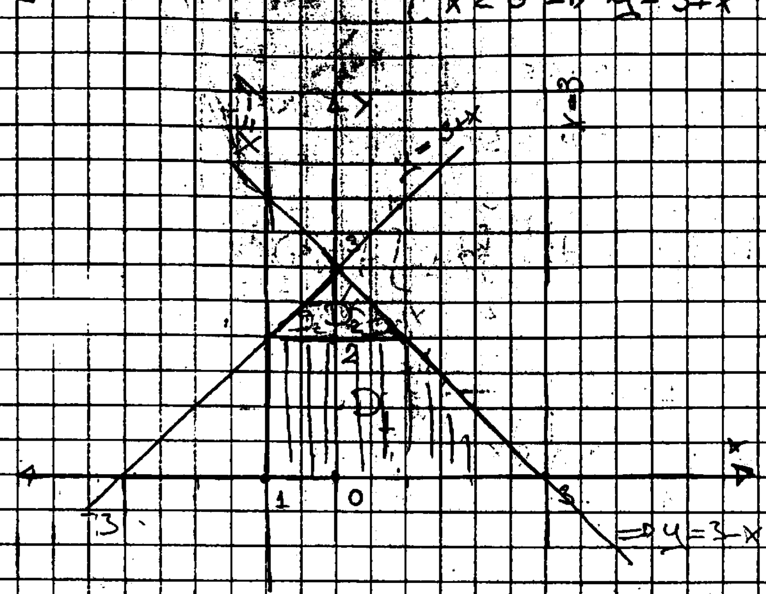
$\boxed{14}$ $\int_{-1}^{3} dx \int_{0}^{3-|x|} f(x,y)\,dy$

$y = 3-|x| = \begin{cases} x>0 \Rightarrow y = 3-x \\ x<0 \Rightarrow y = 3+x \end{cases}$

$D: \begin{cases} -1 \le x \le 3 \\ 0 \le y \le 3-|x| \end{cases}$

$D_1^* = \begin{cases} 0 \le y \le 2 \\ -1 \le x \le 3-y \end{cases}$

$D_2^* = \begin{cases} 2 \le y \le 3 \\ y-3 \le x \le 3-y \end{cases}$

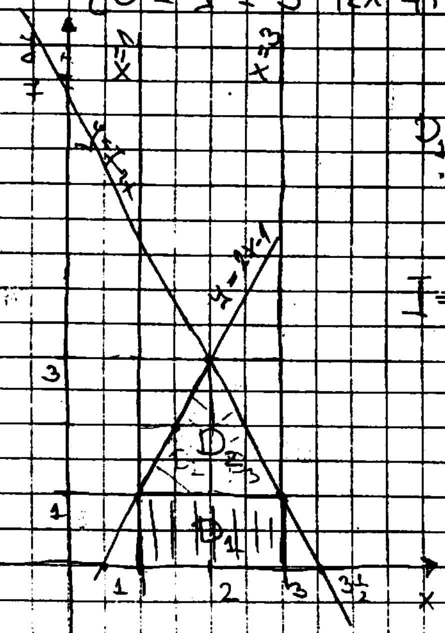


$I = \int_{0}^{2} dy \int_{1}^{3-y} f(x,y)\,dx + \int_{2}^{3} dy \int_{y-3}^{3-y} f(x,y)\,dx$

$\boxed{15}$ $I = \int_{1}^{3} dx \int_{0}^{3-|2x-4|} f(x,y)\,dy$

$D = \begin{cases} 1 \le x \le 3 \\ 0 \le y \le 3-|2x-4| \end{cases}$

$3-|2x-4| = \begin{cases} x>2 \Rightarrow 7-2x = y \\ x<2 \quad 2x-1 = y \end{cases}$



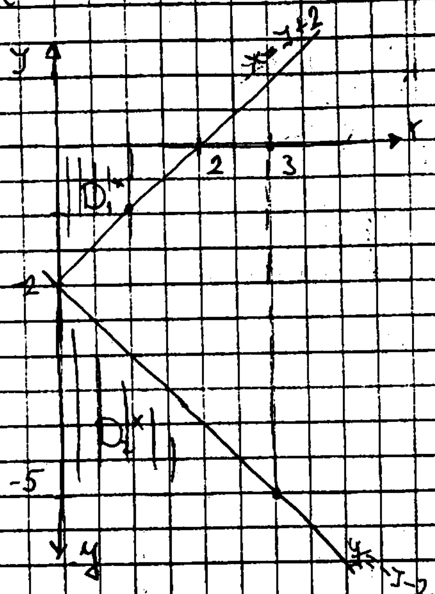
$D_1^* = \begin{cases} 0 \le y \le 1 \\ 1 \le x \le 3 \end{cases}$

$D_2^* = \begin{cases} 1 \le y \le 3 \\ \frac{y+1}{2} \le x \le \frac{7-y}{2} \end{cases}$

$I = \int_{0}^{1} dy \int_{1}^{3} f(x,y)\,dx + \int_{1}^{3} dy \int_{\frac{y+1}{2}}^{\frac{7-y}{2}} f(x,y)\,dx$

16. $I=\int\limits_{-5}^{0} dy \int\limits_{0}^{|y+2|} f(x,y)\,dx$

$$D=\begin{cases} -5 \le y \le 0 \\ 0 \le x \le |y+2| \end{cases} \qquad x=|y+2|=\begin{cases} y+2, & y>-2 \\ -y-2, & y<-2 \end{cases}$$



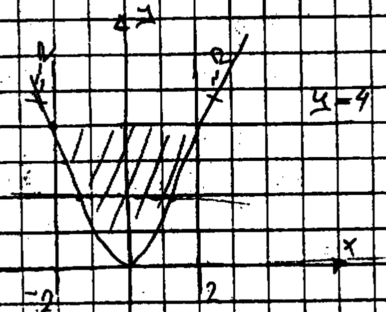
$y=-5$

$$D_1^*=\begin{cases} 0 \le x \le 2 \\ x-2 \le y \le 0 \end{cases}$$

$$D_2^*=\begin{cases} 0 \le x \le 3 \\ -5 \le y \le -x-2 \end{cases}$$

$$I=\int\limits_{0}^{2} dx \int\limits_{x-2}^{0} f(x,y)\,dy+\int\limits_{0}^{3} dx \int\limits_{-5}^{-x-2} f(x,y)\,dy$$

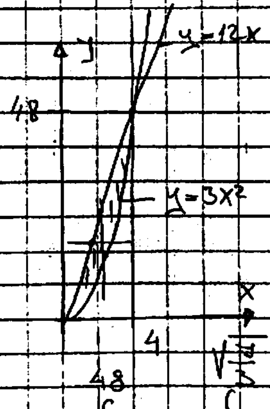
17. $I=\int\limits_{-2}^{2} dx \int\limits_{x^2}^{4} f(x,y)\,dy$



$$D\begin{cases} -2 \le x \le 2 \\ x^2 \le y \le 4 \end{cases}$$

$$D^*=\begin{cases} 0 \le y \le 4 \\ -\sqrt{y} \le x \le \sqrt{y} \end{cases} \qquad I=\int\limits_{0}^{4} dy \int\limits_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\,dx$$

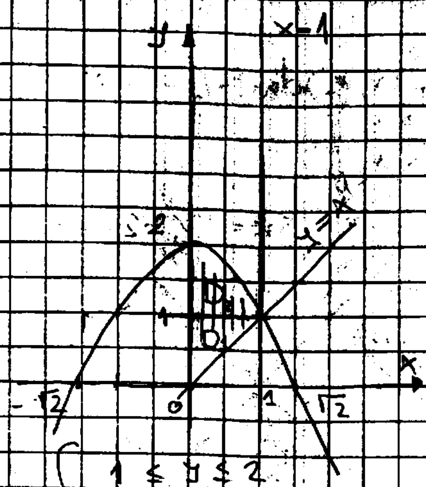
18. $\int\limits_{0}^{4} dx \int\limits_{3x^2}^{12x} f(x,y)\,dy$



$$D=\begin{cases} 0 \le x \le 4 \\ 3x^2 \le y \le 12x \end{cases}$$

$$D^*=\begin{cases} 0 \le y \le 48 \\ \frac{y}{12} \le x \le \sqrt{\frac{y}{3}} \end{cases} \qquad I=\int\limits_{0}^{48} dy \int\limits_{\frac{y}{12}}^{\sqrt{\frac{y}{3}}} f(x,y)\,dx$$

19. $\int_0^1 dx \int_x^{2-x^2} f(x,y)\,dy$

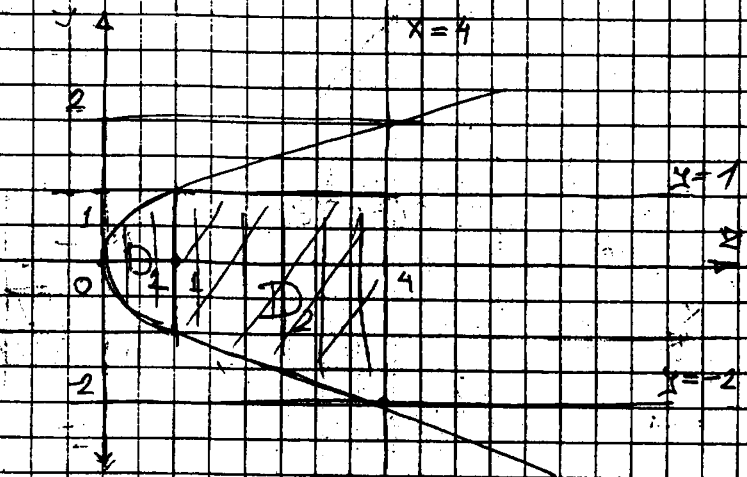


$D: \begin{cases} 0 \le x \le 1 \\ x \le y \le 2-x^2 \end{cases}$

$D_1^* = \begin{cases} 0 \le y \le 1 \\ 0 \le x \le y \end{cases}$ $\quad D_2^* = \begin{cases} 1 \le y \le 2 \\ 0 \le x \le \sqrt{2-y} \end{cases}$

$$I = \int_0^1 dy \int_0^y f(x,y)\,dx + \int_1^2 dy \int_0^{\sqrt{2-y}} f(x,y)\,dx$$

20. $\int_{-2}^1 dy \int_{y^2}^4 f(x,y)\,dx$



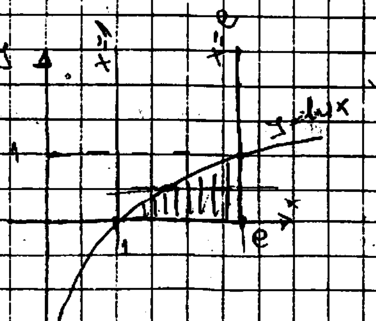
$D = \begin{cases} 2 \le y \le 1 \\ y^2 \le x \le 4 \end{cases}$

$D_1^* = \begin{cases} 0 \le x \le 1 \\ -\sqrt{x} \le y \le \sqrt{x} \end{cases}$

$D_2^* = \begin{cases} 1 \le x \le 4 \\ -\sqrt{x} \le y \le 1 \end{cases}$

$$I = \int_0^1 dx \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dy + \int_1^4 dx \int_{-\sqrt{x}}^{1} f(x,y)\,dy$$

21. $I = \int_1^e dx \int_0^{\ln x} f(x,y)$

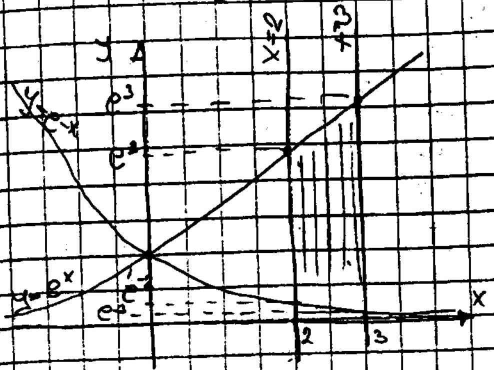


$D^* = \begin{cases} 0 \le y \le 1 \\ e^y \le x \le e \end{cases}$

$D = \begin{cases} 1 \le x \le e \\ 0 \le y \le \ln x \end{cases}$

$$I = \int_0^1 dy \int_{e^y}^{e} f(x,y)\,dx$$

22. $I = \int_2^3 dx \int_{e^{-x}}^{e^x} f(x,y)\, dy$ $\quad D = \begin{cases} 2 \le x \le 3 \\ e^{-x} \le y \le e^x \end{cases}$



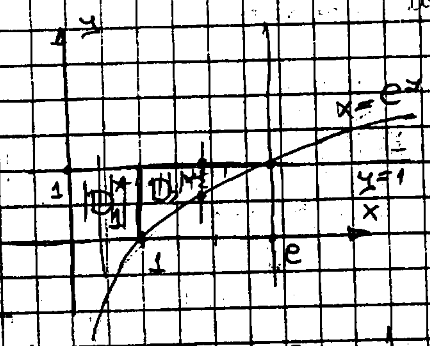
$D_1^* = \begin{cases} e^{-3} \le y \le e^{-2} \\ \ln \frac{1}{y} \le x \le 3 \end{cases}$

$D_2^* = \begin{cases} e^{-2} \le y \le e^2 \\ 2 \le x \le 3 \end{cases}$

$D_3^* = \begin{cases} e^2 \le y \le e^3 \\ \ln y \le x \le 3 \end{cases}$

$$I = \int_{e^{-3}}^{e^{-2}} dy \int_{\ln y^{-1}}^{3} f(x,y)\,dx + \int_{e^{-2}}^{e^2} dy \int_2^3 f(x,y)\,dx + \int_{e^2}^{e^3} dy \int_{\ln y}^{3} f(x,y)\,dx$$

23. $I = \int_0^1 dy \int_0^{e^y} f(x,y)\, dx$



$D = \begin{cases} 0 \le y \le 1 \\ 0 \le x \le e^y \end{cases}$

$D_1^* = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le 1 \end{cases}$

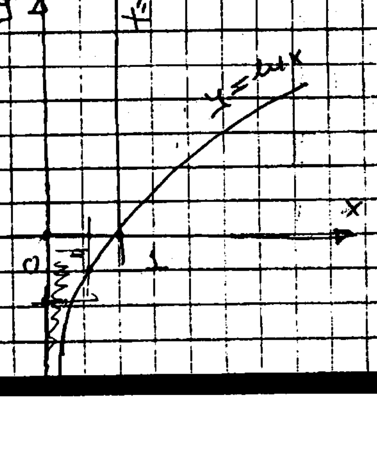
$D_2^* = \begin{cases} 1 \le x \le e \\ \ln x \le y \le 1 \end{cases}$

$$I = \int_0^1 dx \int_0^1 f(x,y)\,dy + \int_1^e dx \int_{\ln x}^1 f(x,y)\,dy$$

24. $\int_0^1 dx \int_{\ln x}^0 f(x,y)\,dy$

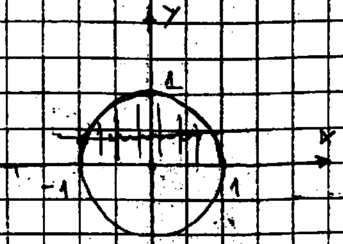
$D = \begin{cases} 0 \le x \le 1 \\ \ln x \le y \le 0 \end{cases}$ $\quad D^* = \begin{cases} -\infty \le y \le 0 \\ 0 \le x \le e^y \end{cases}$

$$\int_{-\infty}^{0} \left(\int_0^{e^y} dx \right) dy$$



25. $I = \int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} f(x,y)\,dx$

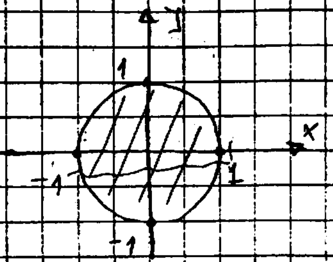
$D^* = \begin{cases} 0 \le y \le 1 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{cases}$



$D = \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \sqrt{1-x^2} \end{cases}$

$I = \int_{0}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$

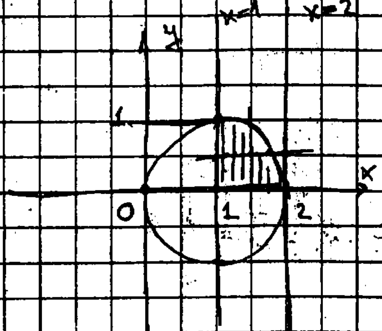
26. $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy$



$D^* = \begin{cases} -1 \le y \le 1 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{cases}$

$I = \int_{-1}^{1} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$

27. $\int_{1}^{2} dx \int_{0}^{\sqrt{2x-x^2}} f(x,y)\,dy$



$y^2 + x^2 - 2x = 0$

$(x-1)^2 + y^2 = 1$

$x - 1 = \pm\sqrt{1-y^2}$

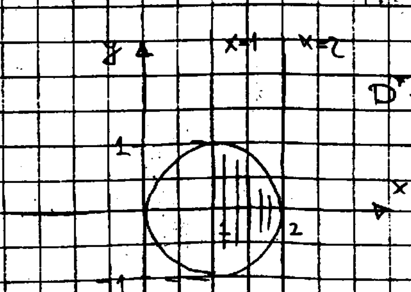
$x = 1 + \sqrt{1-y^2}$

$D = \begin{cases} 1 \le x \le 2 \\ 0 \le y \le \sqrt{2x-x^2} \end{cases}$

$D^* = \begin{cases} 0 \le y \le 1 \\ 1 \le x \le 1 + \sqrt{1-y^2} \end{cases}$

$I = \int_{0}^{1} dy \int_{1}^{1+\sqrt{1-y^2}} f(x,y)\,dx$

28. $\int_{1}^{2} dx \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} f(x,y)\,dy$



$D^* = \begin{cases} -1 \le y \le 1 \\ 1 \le x \le 1+\sqrt{1-y^2} \end{cases}$

$D = \begin{cases} 1 \le x \le 2 \\ -\sqrt{2x-y^2} \le y \le \sqrt{2x-x^2} \end{cases}$

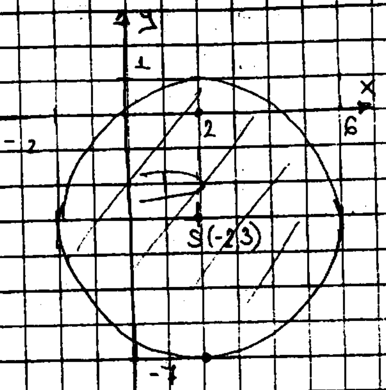
$I = \int_{1}^{1} dy \int_{1}^{1 \pm \sqrt{1-y^2}} f(x,y)\,dx$

129. $$I=\int_{-7}^{1} dy \int_{2-\sqrt{7-6y-y^2}}^{2+\sqrt{7-6y-y^2}} f(x,y)\,dx$$

$$D=\begin{cases} -7 \le y \le 1 \\ 2-\sqrt{7-6y-y^2} \le x \le 2+\sqrt{7-6y-y^2} \end{cases}$$

$x = 2-\sqrt{7-6y-y^2}$ $\quad y=-3\pm\sqrt{12-x^2+4x}$

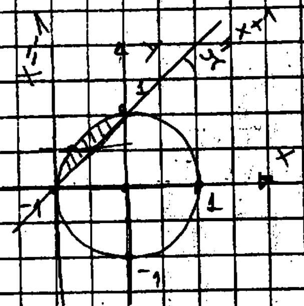
$(x-2)^2+(y+3)^2=16$



$$D^*=\begin{cases} -2 \le x \le 6 \\ -3-\sqrt{} \le y \le -3+\sqrt{12-x^2+4x} \end{cases}$$

$$I=\int_{-2}^{6} dx \int_{-3-\sqrt{12-x^2+4x}}^{-3+\sqrt{12-x^2+4x}} f(x,y)\,dy$$

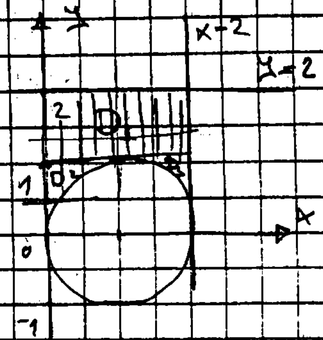
130. $$\int_{-1}^{0} dx \int_{x+1}^{\sqrt{1-x^2}} f(x,y)\,dy$$



$$D^*=\begin{cases} 0 \le y \le 1 \\ -\sqrt{1-y^2} \le x \le y-1 \end{cases}$$

$$D=\begin{cases} -1 \le x \le 0 \\ x+1 \le y \le \sqrt{1-x^2} \end{cases}$$

$$I=\int_{0}^{1} dy \int_{-\sqrt{1-y^2}}^{y-1} f(x,y)\,dx$$



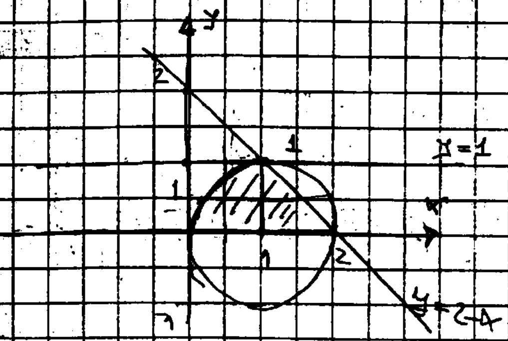
131. $$\int_{0}^{2} dx \int_{\sqrt{2x-x^2}}^{2} f(x,y)$$ $\quad D: \begin{cases} 0 \le x \le 2 \\ \sqrt{2x-x^2} \le y \le 2 \end{cases}$

$$D_1^*=\begin{cases} 1 \le y \le 2 \\ 0 \le x \le 2 \end{cases} \qquad D_2^*=\begin{cases} 0 \le y \le 1 \\ 1 \le x \le 1-\sqrt{1-y^2} \end{cases}$$

$$D_3^*=\begin{cases} 0 \le y \le 1 \\ 1+\sqrt{1-y^2} \le x \le 2 \end{cases}$$

$$I=\int_{0}^{1} dy \int_{1+\sqrt{1-y^2}}^{2} f(x,y)\,dx+\int_{0}^{1} dy \int_{1}^{1-\sqrt{1-y^2}} f(x,y)\,dx+\int_{1}^{2} dy \int_{0}^{2} dx$$

32. $\int_0^1 dy \int_{1-\sqrt{1-y^2}}^{2-y} f(x,y)\,dx$



$D = \begin{cases} 0 \le y \le 1 \\ 1-\sqrt{1-y^2} \le x \le 2-y \end{cases}$

$x = 1-\sqrt{1-y^2}$

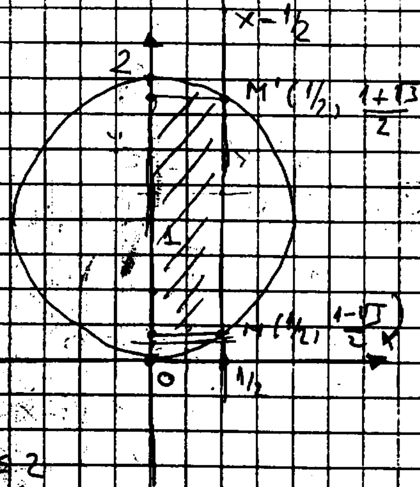
$(x-1)^2+y^2=1$

$y = \pm\sqrt{2x-x^2}$

$D_1^* = \begin{cases} 0 \le x \le 1 \\ 0 \le y \le \sqrt{2x-x^2} \end{cases}$ $\qquad D_2^* = \begin{cases} 1 \le x \le 2 \\ 0 \le y \le 2-x \end{cases}$

$I = \int_0^1 dx \int_0^{\sqrt{2x-x^2}} f(x,y)\,dy + \int_1^2 dx \int_0^{2-x} f(x,y)\,dy$

33. $\int_0^{1/2} dx \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} f(x,y)\,dy$



$D = \begin{cases} 0 \le x \le 1/2 \\ 1-\sqrt{1-x^2} \le y \le 1+\sqrt{1-x^2} \end{cases}$

$y = 1 \pm \sqrt{1-x^2}$

$(y-1)^2+x^2=1$

$x = \pm\sqrt{2y-y^2}$

$D_1^* = \begin{cases} 0 \le y \le \frac{1-\sqrt{3}}{2} \\ 0 \le x \le \sqrt{2y-y^2} \end{cases}$ $\qquad D_3^* = \begin{cases} \frac{1+\sqrt{3}}{2} \le y \le 2 \\ 0 \le x \le \sqrt{2y-y^2} \end{cases}$

$D_2^* = \begin{cases} \frac{1-\sqrt{3}}{2} \le y \le \frac{1+\sqrt{3}}{2} \\ 0 \le x \le 1/2 \end{cases}$

$I = \int_0^{\frac{1-\sqrt{3}}{2}} dy \int_0^{\sqrt{2y-y^2}} f(x,y)\,dx + \int_{\frac{1-\sqrt{3}}{2}}^{\frac{1+\sqrt{3}}{2}} dy \int_0^{1/2} f(x,y)\,dx + \int_{\frac{1+\sqrt{3}}{2}}^{2} dy \int_0^{\sqrt{2y-y^2}} f(x,y)\,dx$

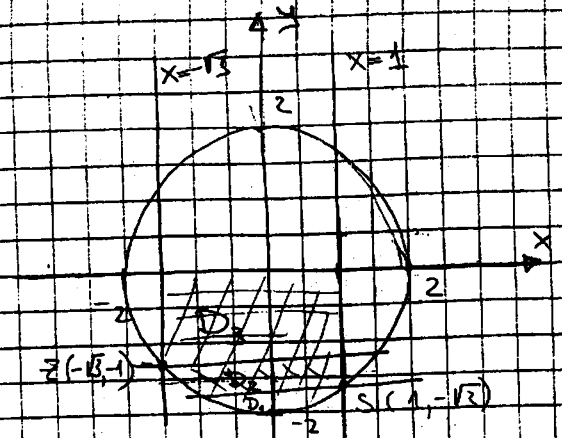
34. $\int_{-\sqrt{3}}^{1} dx \int_{-\sqrt{4-x^2}}^{0} u(x,y)\,dy$

$y = -\sqrt{4-x^2}$

$k:\ y^2+x^2=4$

$D = \begin{cases} -\sqrt{3} \le x \le 1 \\ -\sqrt{4-x^2} \le y \le 0 \end{cases}$

$x = \pm\sqrt{4-y^2}$



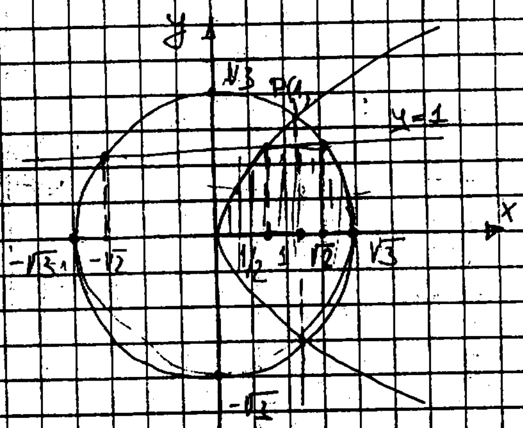
$$D_1^* = \begin{cases} -2 \le y \le -\sqrt{3} \\ -\sqrt{4-y^2} \le x \le \sqrt{4-y^2} \end{cases}$$

$$D_2^* = \begin{cases} -\sqrt{3} \le y \le -1 \\ -\sqrt{4-y^2} \le x \le 1 \end{cases}$$

$$D_3^* = \begin{cases} -1 \le y \le 0 \\ -\sqrt{3} \le x \le 1 \end{cases}$$

$$I = \int_{-2}^{-\sqrt{3}} dy \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} f(x,y)\,dx + \int_{-\sqrt{3}}^{-1} dy \int_{-\sqrt{4-y^2}}^{1} f(x,y)\,dx + \int_{-1}^{0} dy \int_{-\sqrt{3}}^{1} f(x,y)\,dx$$

$\boxed{135.}$ $\displaystyle\int_0^1 dy \int_{y^2/2}^{\sqrt{3-y^2}} f(x,y)\,dx$



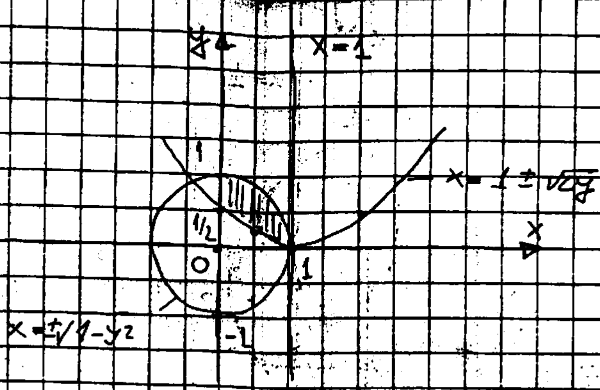
$$D = \begin{cases} 0 \le y \le 1 \\ \frac{y^2}{2} \le x \le \sqrt{3-y^2} \end{cases}$$

$$D_1^* = \begin{cases} 0 \le x \le 1/2 \\ 0 \le y \le \sqrt{2x} \end{cases}$$

$$D_2^* = \begin{cases} 1/2 \le x \le \sqrt{2} \\ 0 \le y \le 1 \end{cases} \qquad D_3^* = \begin{cases} \sqrt{2} \le x \le \sqrt{3} \\ 0 \le y \le \sqrt{3-x^2} \end{cases}$$

$$I = \int_0^{1/2} dx \int_0^{\sqrt{2x}} f(x,y)\,dy + \int_{1/2}^{\sqrt{2}} dx \int_0^{1} f(x,y)\,dy + \int_{\sqrt{2}}^{\sqrt{3}} dx \int_0^{\sqrt{3-x^2}} f(x,y)\,dy$$

36. $\displaystyle\int_0^1 dx \int_{\frac{(1-x)^2}{2}}^{\sqrt{1-x^2}} f(x,y)\,dy$



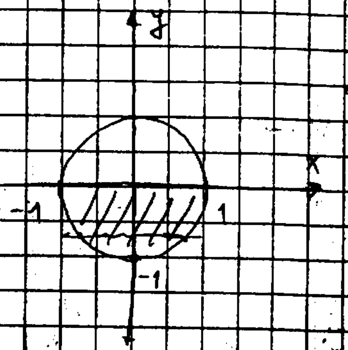
$$D = \begin{cases} 0 \le x \le 1 \\ \frac{(1-x)^2}{2} \le y \le \sqrt{1-x^2} \end{cases}$$

$$D_1^* = \begin{cases} 0 \le y \le 1/2 \\ 1-\sqrt{2y} \le x \ \sqrt{1-y^2} \end{cases}$$

$$D_2^* = \begin{cases} 1/2 \le y \le 1 \\ 0 \le x \le \sqrt{1-y^2} \end{cases}$$

$$I = \int_0^{1/2} dy \int_{\sqrt{1-2y}}^{\sqrt{1-y^2}} f(x,y)\,dx + \int_{1/2}^{1} dy \int_0^{\sqrt{1-y^2}} f(x,y)\,dx$$

37. $\displaystyle\int_{-1}^1 dx \int_{-\sqrt{1-x^2}}^{0} f(x,y)\,dy$

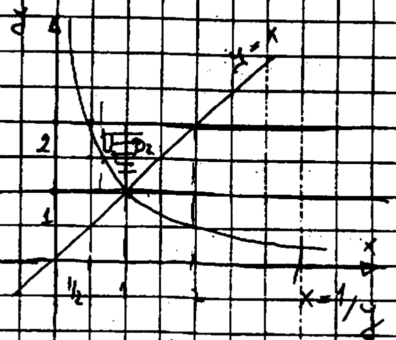


$$D^* = \begin{cases} -1 \le y \le 0 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{cases}$$

$$D = \begin{cases} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le 0 \end{cases}$$

$$I = \int_{-1}^{0} dy \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y)\,dx$$

38. $\displaystyle\int_1^2 dy \int_{1/y}^{y} f(x,y)\,dx$

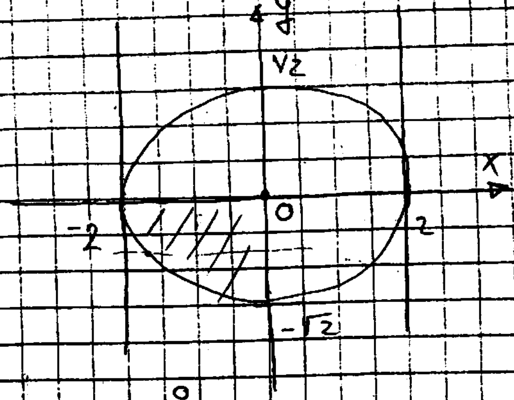


$$D = \begin{cases} 1 \le y \le 2 \\ \frac{1}{y} \le x \le y \end{cases}$$

$$D_1^* = \begin{cases} 1/2 \le x \le 1 \\ \frac{1}{x} \le y \le 2 \end{cases} \qquad D_2^* = \begin{cases} 1 \le x \le 2 \\ x \le y \le 2 \end{cases}$$

$$I = \int_{1/2}^{1} dx \int_{1/x}^{2} f(x,y)\,dy + \int_1^2 dx \int_x^2 f(x,y)\,dy$$

39. $\int_{-2}^{0} dx \int_{-\sqrt{2-(x^2/2)}}^{0} f(x,y)\,dy$



$y = -\sqrt{2-(x^2/2)}$

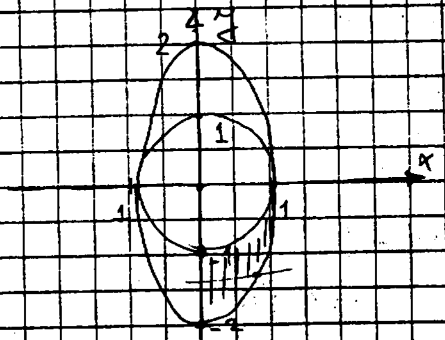
$y^2 + \frac{x^2}{2} = 2$

$\frac{y^2}{2} + \frac{x^2}{4} = 1$

$D = \begin{cases} -2 \le x \le 0 \\ -\sqrt{2-(x^2/2)} \le y \le 0 \end{cases}$

$D^* = \begin{cases} -\sqrt{2} \le y \le 0 \\ -\sqrt{4-2y^2} \le x \le 0 \end{cases}$ $\quad I = \int_{-\sqrt{2}}^{0} dy \int_{-\sqrt{4-2y^2}}^{0} f(x,y)\,dx$

40. $\int_{0}^{1} dx \int_{-2\sqrt{1-x^2}}^{-\sqrt{1-x^2}} f(x,y)\,dy$



$D = \begin{cases} 0 \le x \le 1 \\ -2\sqrt{1-x^2} \le y \le -\sqrt{1-x^2} \end{cases}$

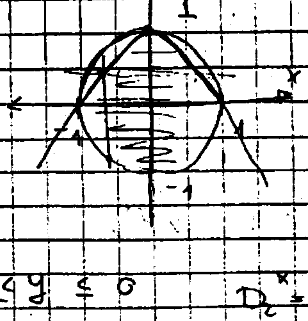
$y = -2\sqrt{1-x^2} \Rightarrow x^2 + \frac{y^2}{4} = 1 \quad x = \pm \frac{\sqrt{4-y^2}}{2}$

$y = -\sqrt{1-x^2} \Rightarrow x^2 + y^2 = 1 \quad x = \pm\sqrt{1-y^2}$

$D_1^* = \begin{cases} -2 \le y \le -1 \\ 0 \le x \le \frac{\sqrt{4-y^2}}{2} \end{cases}$ $\quad D_2^* = \begin{cases} -1 \le y \le 0 \\ \sqrt{1-y^2} \le x \le \frac{\sqrt{4-y^2}}{2} \end{cases}$

$I = \int_{-2}^{-1} dy \int_{0}^{\frac{1}{2}\sqrt{4-y^2}} f(x,y)\,dx + \int_{-1}^{0} dy \int_{\sqrt{1-y^2}}^{\frac{1}{2}\sqrt{4-y^2}} f(x,y)\,dx$

41. $\int_{-1}^{1} dx \int_{-\sqrt{1-x^2}}^{1-x^2} f(x,y)\,dy$



$y = \frac{1}{2}$

$\frac{1}{4} + x^2 = 1 \quad x = \frac{\sqrt{3}}{2} = 0{,}8$

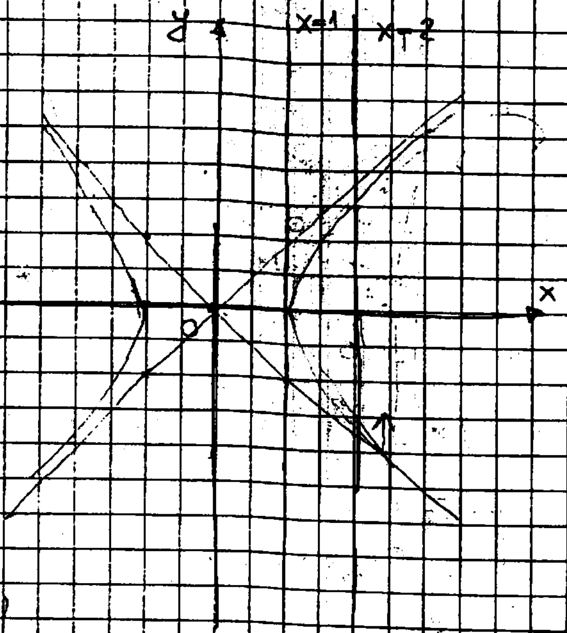
$y = \frac{1}{2} \quad x = \sqrt{1-\frac{1}{2}} = \frac{1}{\sqrt{2}}$

$D = \begin{cases} -1 \le x \le 1 \\ -\sqrt{1-x^2} \le y \le 1-x^2 \end{cases}$

$y = -\sqrt{1-x^2} \Rightarrow y^2 + x^2 = 1 \Rightarrow x = \pm\sqrt{1-y^2}$

$y = 1-x^2 \Rightarrow x = \pm\sqrt{1-y}$ $\quad D_1^* = \begin{cases} -1 \le y \le 0 \\ -\sqrt{1-y^2} \le x \le \sqrt{1-y^2} \end{cases}$ $\quad D_2^* = \begin{cases} 0 \le y \le 1 \\ -\sqrt{1-y} \le x \le \sqrt{1-y} \end{cases}$

42. $\int_1^2 dx \int_{-\sqrt{x^2-1}}^{\sqrt{x^2-1}} f(x,y)\,dy$



$$D = \begin{cases} 1 \le x \le 2 \\ -\sqrt{x^2-1} \le y \le \sqrt{x^2-1} \end{cases}$$

$y = \pm\sqrt{x^2-1}$ $\quad y\cdot$

$y^2 = x^2 - 1$

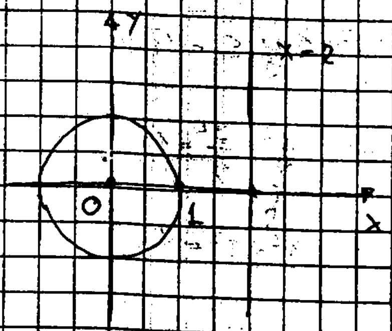
$\boxed{x^2 - y^2 = 1}$

$-\sqrt{x^2-1} \le y \le \sqrt{x^2-1}$

$y \geq -\sqrt{x^2-1} \Rightarrow y>0 \lor y<0 \quad y^2 \le x^2-1$

$y<0 \quad x^2-y^2 \geq 1$

43. $\int_0^2 dx \int_{-\sqrt{1-x^2}}^{\sqrt{1+x^2}} f(x,y)\,dy$



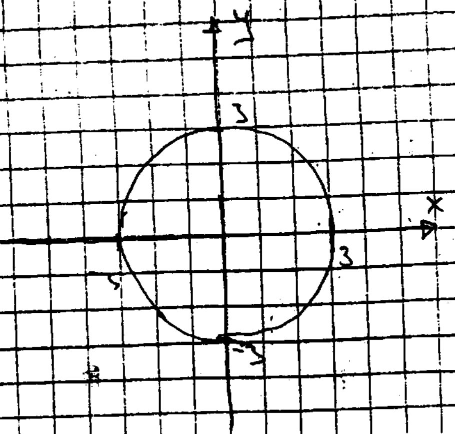
$$D = \begin{cases} 0 \le x \le 2 \\ -\sqrt{1+x^2} \le y \le \sqrt{1+x^2} \end{cases}$$

44. $\int_{-2}^0 dx \int_{\sqrt{1+x^2}}^{\sqrt{9-x^2}} f(x;y)\,dy$

$$D = \begin{cases} -2 \le x \le 0 \\ \sqrt{1+x^2} \le y \le \sqrt{9-x^2} \end{cases}$$

$y = \sqrt{1+x^2} \Rightarrow \boxed{y^2 - x^2 = 1} \to x = \pm\sqrt{y^2}$

$y = \sqrt{9-x^2} \Rightarrow \boxed{x^2+y^2=9} \to x = \pm\sqrt{9-y^2}$

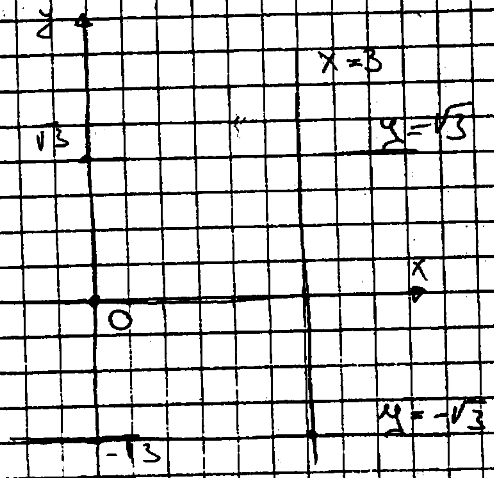


45. $\int_{-\sqrt{3}}^{\sqrt{3}} dy \int_{1+\sqrt{1+y^2}}^{3} f(x,y)\,dx$

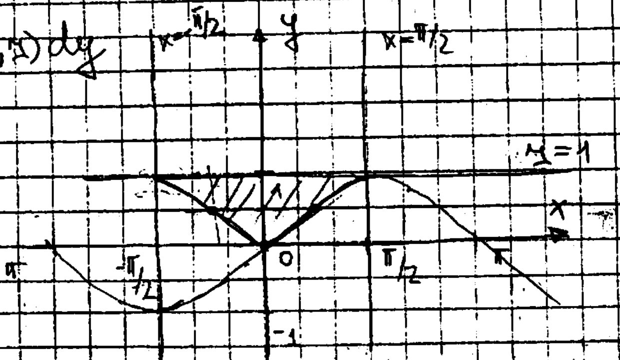
$D = \begin{cases} -\sqrt{3} \le y \le \sqrt{3} \\ 1+\sqrt{1+y^2} \le x \le 3 \end{cases}$

$x = 1 + \sqrt{1+y^2}$

$(x-1)^2 - y^2 = 1 \Rightarrow y = \pm\sqrt{x^2-2x}$



46. $\int_{-\pi/2}^{\pi/2} dx \int_{|\sin x|}^{1} f(x,y)\,dy$



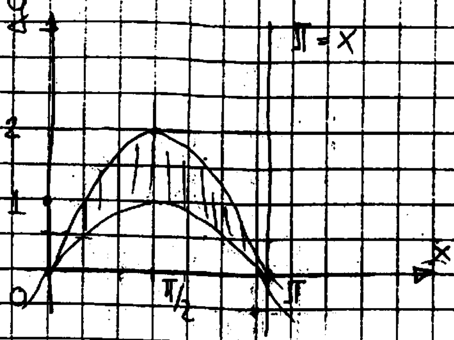
$D = \begin{cases} -\pi/2 \le x \le \pi/2 \\ |\sin x| \le y \le 1 \end{cases}$

$D^* = \begin{cases} 0 \le y \le 1 \\ \arcsin(-y) \le x \le \arcsin(y) \end{cases}$

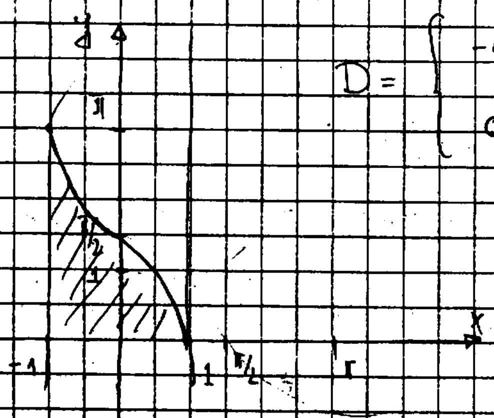
$I = \int_0^1 dy \int_{\arcsin(-y)}^{\arcsin y} f(x,y)\,dx$

47. $\displaystyle\int_0^{\pi} dx \int_{\sin x}^{2\sin x} f(x,y)\,dy$

$$D = \begin{cases} 0 \le x \le \pi \\ \sin x \le y \le 2\sin x \end{cases}$$



48. $\displaystyle\int_{-1}^{1} dx \int_{0}^{\arccos x} f(x,y)\,dy$

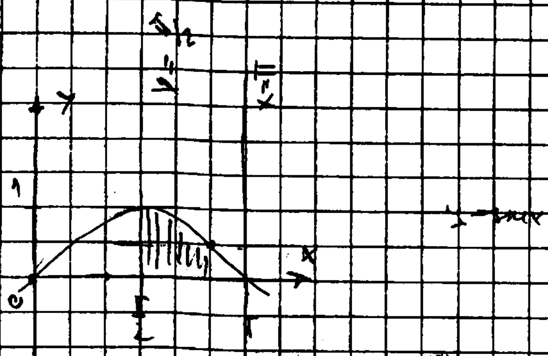


$$D = \begin{cases} -1 \le x \le 1 \\ 0 \le y \le \arccos x \end{cases}$$

$$D^* = \begin{cases} 0 \le y \le \pi \\ -1 \le x \le \cos y \end{cases}$$

$$I = \int_0^{\pi} dy \int_{-1}^{\cos y} f(x,y)\,dx$$

49. $\displaystyle\int_{\pi/2}^{\pi} dx \int_0^{\sin x} f(x,y)\,dy$

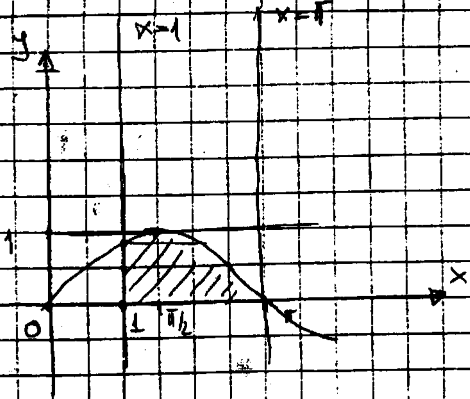


$$D = \begin{cases} \pi/2 \le x \le \pi \\ 0 \le y \le \sin x \end{cases}$$

$$D^* = \begin{cases} 0 \le y \le 1 \\ \pi/2 \le x \le \arcsin y \end{cases}$$

$$I = \int_0^{1} dy \int_{\pi/2}^{\arcsin y} f(x,y)\,dx$$

50. $\int_1^{\pi} dx \int_0^{\sin x} f(x,y)\,dy$

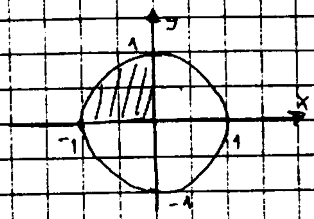


$$D = \begin{cases} 1 \le x \le \pi \\ 0 \le y \le \sin x \end{cases}$$

6.4. Смена променљивих у двојном интегралу

А) Параметризовати област D у Декартовој равни уводећи поларне координате $x=\rho\cos\varphi$, $y=\rho\sin\varphi$

1.) $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1 \ \wedge\ x\le 0 \ \wedge\ y\ge 0\}$



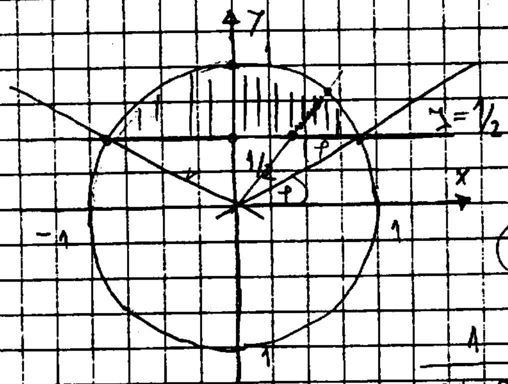
$\left.\begin{array}{l} x=\rho\cos\varphi \\ y=\rho\sin\varphi \end{array}\right\}$ $k: \underline{\rho=1}$

k = кружница (то је уједно и највеће ρ тј. r кружнице је $\rho_{max}=1$)

$0\le\rho\le 1$

$\frac{\pi}{2}\le\varphi\le\pi$

2.) $D: \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1 \quad y\ge\frac{1}{2}\}$



$k: \rho=1$

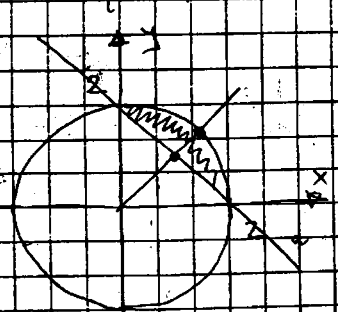
$y=\rho\sin\varphi \rightarrow \frac{1}{2}=\rho\sin\varphi \rightarrow \boxed{\rho=\frac{1}{2\sin\varphi}}$

$(*\ \sin\varphi=\frac{1}{2} \quad \varphi=\frac{\pi}{6})$

$\frac{1}{2\sin\varphi}\le\rho\le 1$

$\frac{\pi}{6}\le\varphi\le\frac{5\pi}{6}$

3.) $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 4 \quad x+y-2\ge 0\}$



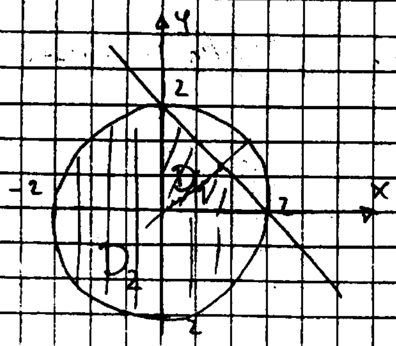
$k: \rho=2$

$a: \rho=\frac{2}{\cos\varphi+\sin\varphi}$

$\frac{2}{\cos\varphi+\sin\varphi}\le\rho\le 2$

$0\le\varphi\le\pi/2$

4. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4 \ \wedge\ x+y-2 \le 0\}$



$k: \rho = 2$

$a: \rho = \dfrac{2}{\cos\varphi + \sin\varphi}$

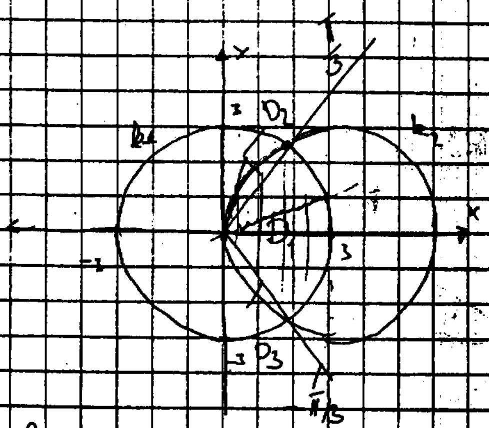
$D_1: \begin{cases} 0 \le \rho \le \dfrac{2}{\cos\varphi+\sin\varphi} \\ 0 \le \varphi \le \pi/2 \end{cases}$

$D_2: \begin{cases} 0 \le \rho \le 2 \\ \pi/2 \le \varphi \le 2\pi \end{cases}$

5. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 9 \ \wedge\ x^2+y^2 \le 6x\}$

$k_1: \rho = 3$

$k_2: \rho = 6\cos\varphi$



$k_1 \cap k_2 = \cancel{x^2}+6x-\cancel{x^2}=9$

$x = \frac{9}{6} = \frac{3}{2}$

$\frac{3}{2} = 3\cos\varphi \Rightarrow$

$\cos\varphi = \frac{1}{2} \Rightarrow$

$\varphi = \frac{\pi}{3}$

$D_1 = \begin{cases} 0 \le \rho \le 3 \\ -\pi/3 \le \varphi \le \pi/3 \end{cases}$

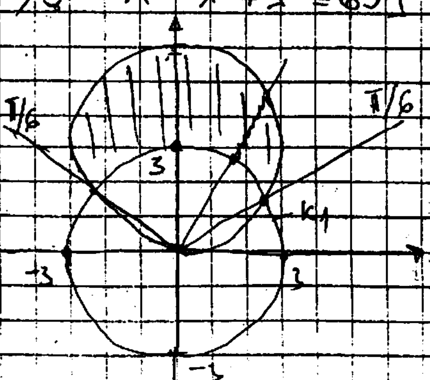
$D_2 = \begin{cases} 0 \le \rho \le 6\cos\varphi \\ \pi/3 \le \varphi \le \pi/2 \end{cases}$

$D_3 = \begin{cases} 0 \le \rho \le 6\cos\varphi \\ -\pi/3 \le \varphi \le -\pi/2 \end{cases}$

6. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \ge 9 \ \wedge\ x^2+y^2 \le 6y\}$

$k_1: \rho = 3$

$k_2: \rho = 6\sin\varphi$



$k_1 \cap k_2 \Rightarrow$

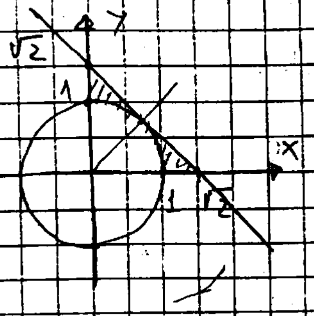
$3 = 6\sin\varphi$

$\sin\varphi = \frac{1}{2}$

$\varphi = \frac{\pi}{6}$

$D = \begin{cases} 3 \le \rho \le 6\sin\varphi \\ \pi/6 \le \varphi \le \frac{5\pi}{6} \end{cases}$

7. $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \geq 1 \ \wedge \ x+y \leq \sqrt{2} \}$

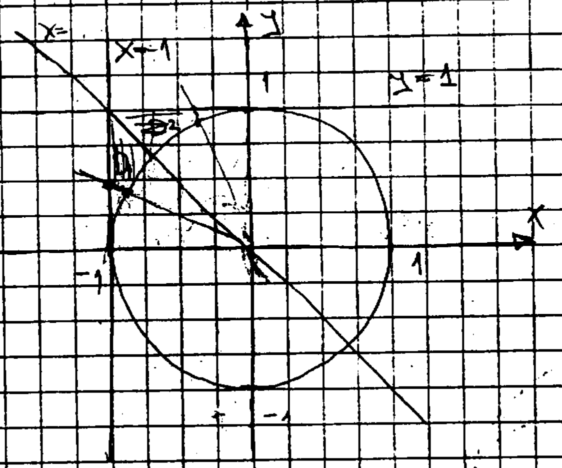


$k_1: \ \rho = 1$

$b: \ \rho = \dfrac{\sqrt{2}}{\cos\varphi + \sin\varphi}$

$$D = \begin{cases} 1 \leq \rho \leq \dfrac{\sqrt{2}}{\cos\varphi + \sin\varphi} \\ 0 \leq \varphi \leq \pi/2 \end{cases}$$

8. $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \geq 1 \ \wedge \ -1 \leq x \leq 0 \ \wedge \ 0 \leq y \leq 1 \}$



$k_1: \ \rho = 1$

$p_1: \ x = -1 \quad -1 = \rho\cos\varphi$

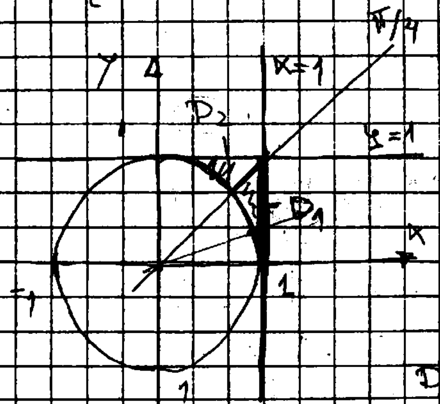
$\rho = -\dfrac{1}{\cos\varphi}$

$p_2 \Rightarrow y = 1 \quad 1 = \rho\sin\varphi$

$\rho = \dfrac{1}{\sin\varphi}$

$$D_1: \begin{cases} 1 \leq \rho \leq -\dfrac{1}{\cos\varphi} \\ \dfrac{[illegible]}{4} \leq \varphi \leq \end{cases} \qquad D_2: \begin{cases} 1 \leq \rho \leq \dfrac{1}{\sin\varphi} \\ \leq \varphi \leq \end{cases}$$

9. $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \geq 1 \ \wedge \ 0 \leq x \leq 1 \ \wedge \ 0 \leq y \leq 1 \}$



$k: \ \rho = 1$

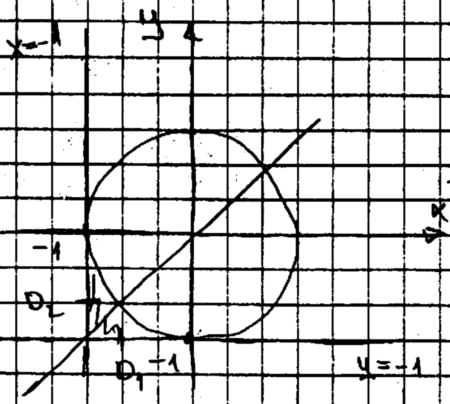
$x = 1 \Rightarrow \rho = \dfrac{1}{\cos\varphi}$

$y = 1 \Rightarrow \rho = \dfrac{1}{\sin\varphi}$

$$D_1: \begin{cases} 1 \leq \rho \leq \dfrac{1}{\cos\varphi} \\ 0 \leq \varphi \leq \pi/4 \end{cases}$$

$$D_2: \begin{cases} 1 \leq \rho \leq \dfrac{1}{\sin\varphi} \\ \pi/4 \leq \varphi \leq \pi/2 \end{cases}$$

10. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \ge 1 \wedge -1 \le x \le 0 \wedge -1 \le y \le 0\}$



$K: \rho = 1$

$x = -1: \quad \rho = -\frac{1}{\cos\varphi}$

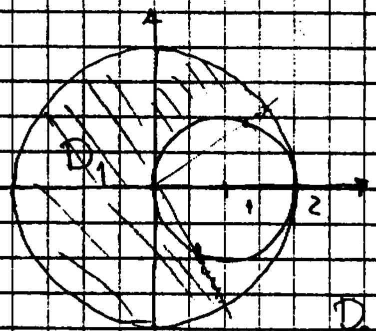
$y = -1: \quad \rho = \frac{1}{\sin\varphi}$

$D_1 = \begin{cases} 1 \le \rho \le -\frac{1}{\sin\varphi} \\ \quad \le \varphi \le \end{cases}$ $\quad D_2 = \begin{cases} 1 \le \rho \le -\frac{1}{\cos\varphi} \\ \quad \le \varphi \le \end{cases}$

11. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 4 \wedge x^2+y^2 \ge 2x\}$

$k_1: \rho = 2$

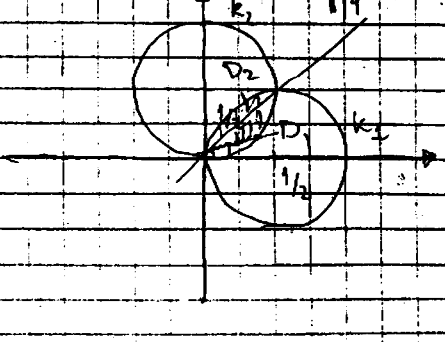
$k_2: \rho = 2\cos\varphi$



$D_1: \begin{cases} 0 \le \rho \le 2 \\ \pi/2 \le \varphi \le \frac{3\pi}{2} \end{cases}$ $\quad D_2: \begin{cases} 2\cos\varphi \le \rho \le 2 \\ -\pi/2 \le \varphi \le \frac{\pi}{2} \end{cases}$

12. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le x \wedge x^2+y^2 \le y\}$

$\left.\begin{matrix} k_1: \rho = \cos\varphi \\ k_2: \rho = \sin\varphi \end{matrix}\right\} k_1 \cap k_2 \Rightarrow \cos\varphi = \sin\varphi \Rightarrow \varphi = \pi/4$



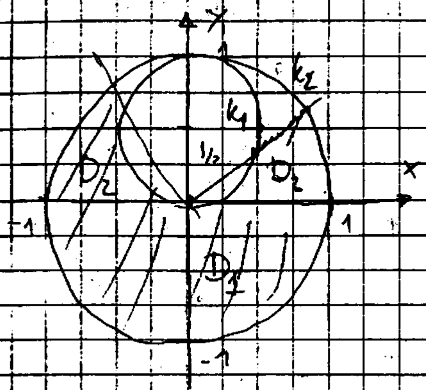
$D_1: \begin{cases} 0 \le \rho \le \sin\varphi \\ 0 \le \varphi \le \pi/4 \end{cases}$

$D_2: \begin{cases} 0 \le \rho \le \cos\varphi \\ \pi/4 \le \varphi \le \pi/2 \end{cases}$

13. $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1 \ \wedge\ x^2+y^2\ge x\}$

$k_1:\ \rho=1$

$k_2:\ \rho=\sin\varphi$



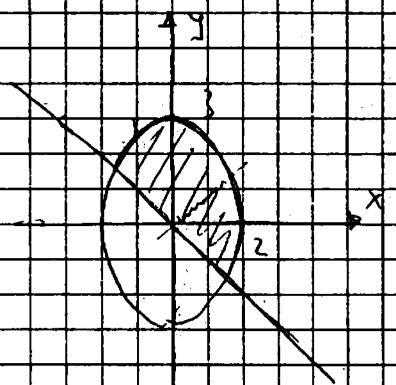
$D_1\ \begin{cases} 0\le\rho\le 1 \\ \pi\le\varphi\le 2\pi \end{cases}$

$D_2:\ \begin{cases} 1\le\rho\le\sin\varphi \\ 0\le\varphi\le\pi \end{cases}$

14.

$D=\{(x,y)\in\mathbb{R}^2 \mid 9x^2+4y^2\le 36 \ \wedge\ x+y\ge 0\}$

$\frac{x^2}{4}+\frac{y^2}{9}=1 \Rightarrow$ $\left.\begin{array}{l} x=2\cos\varphi \\ y=3\cos\varphi \end{array}\right\}$ $e:\ (\rho=1)$

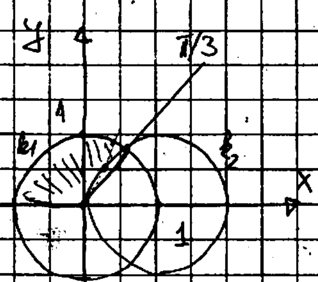


$D=\begin{cases} 0\le\rho\le 1 \\ \quad\le\varphi\le \end{cases}$

17. $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 1 \ \wedge\ x^2+y^2\ge 2x \ \wedge\ y\ge 0\}$

$k_1:\ \rho=1$

$k_2\ \ \rho=2\cos\varphi$



$k_1\cap k_2 \Rightarrow 1=2\cos\varphi$

$\varphi=\frac{\pi}{3}$

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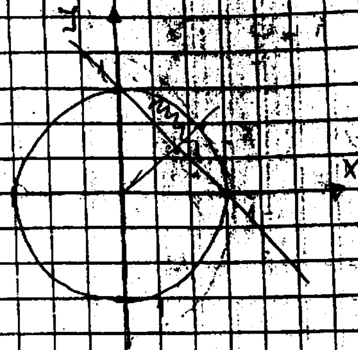
$D_1=\begin{cases} 2\cos\varphi\le\rho\le 1 \\ \pi/3\le\varphi\le\pi/2 \end{cases}$

$D_2=\begin{cases} 0\le\rho\le 1 \\ \pi/2\le\varphi\le\pi \end{cases}$

18. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 1 \wedge x+y-1 \ge 0\}$

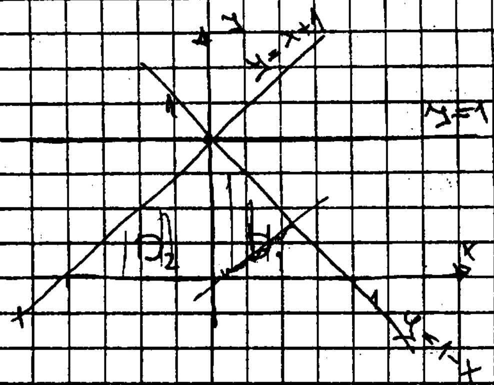
$k_1: \rho = 1$

$p: \rho = \dfrac{1}{\cos\varphi + \sin\varphi}$



$$\left.\begin{array}{l} \dfrac{1}{\cos\varphi+\sin\varphi} \le \rho \le 1 \\ 0 \le \varphi \le \pi/2 \end{array}\right\} = D$$

19. $D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \wedge y-1 \le x \le 1-y\}$



$p_1: y = 1-x \Rightarrow \rho = \dfrac{1}{\cos\varphi+\sin\varphi}$

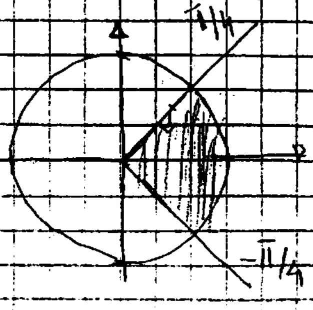
$p_2: y = x+1 \Rightarrow \rho = \dfrac{1}{\sin\varphi - \cos\varphi}$

$D_1: \begin{cases} 0 \le \rho \le \dfrac{1}{\cos\varphi+\sin\varphi} \\ 0 \le \varphi \le \pi/2 \end{cases}$

$D_2: \begin{cases} 0 \le \rho \le \dfrac{1}{\sin\varphi-\cos\varphi} \\ \pi/2 \le \varphi \le \pi \end{cases}$

20. $D = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le 9 \quad x-|y| \ge 0$ $\qquad |y| \le x$

$k_1: x^2+y^2 = 9 \Rightarrow \rho = 3$ $\qquad x = |y| = \begin{cases} y, & y \ge 0 \\ -y, & y < 0 \end{cases}$ $\qquad y \le x \quad -y \le x \quad y \le x$



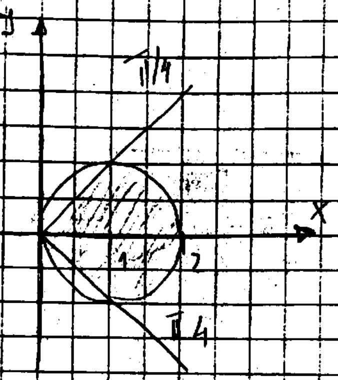
$D = \begin{cases} 0 \le \rho \le 3 \\ -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4} \end{cases}$

$x - |y| \ge 0 \iff |y| \le x \iff x \ge 0$

$y \le x \wedge y \ge 0$

$-y \le x \wedge y < 0$

21. $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le 2x \ \wedge\ x-|y|\ge 0\}$



$x = 1+\rho\cos\varphi$

$y = \rho\sin\varphi$

$\rho^2 \le 2\rho\cos\varphi \Rightarrow \rho\le 2\cos\varphi$

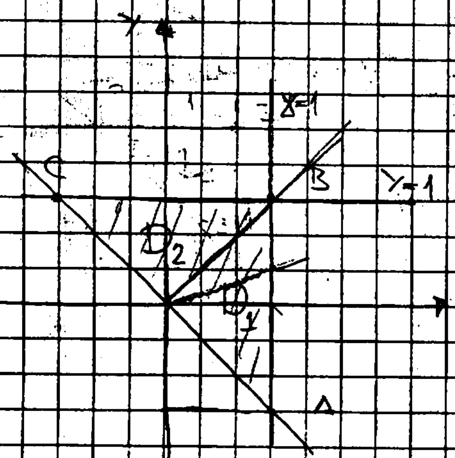
$\varphi \quad \rho$

22. $A(1,-1)$, $B(1,1)$, $C(-1,1)$

$AB \Rightarrow x=1 \rightarrow \rho=\dfrac{1}{\cos\varphi}$

$CB \Rightarrow y=1 \rightarrow \rho=\dfrac{1}{\sin\varphi}$

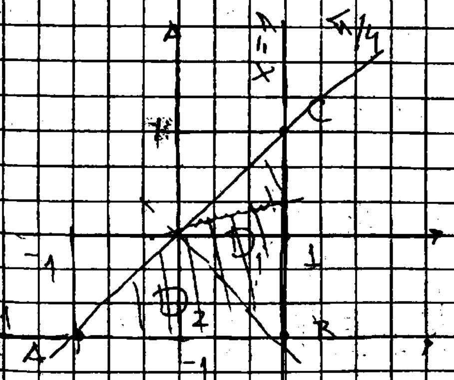
$AC \rightarrow \underline{y=-x}$



$D_1=\begin{cases} 0\le\rho\le\dfrac{1}{\cos\varphi} \\ -\pi/4\le\varphi\le\pi/4\end{cases}$

$D_2=\begin{cases} 0\le\rho\le\dfrac{1}{\sin\varphi} \\ \pi/4\le\varphi\le\dfrac{3\pi}{4}\end{cases}$

23. $A(-1,-1)$ $B(1,-1)$, $C(1,1)$



$\underline{AC}: y=x$

$\underline{BC}: x=1 \Rightarrow \rho=\dfrac{1}{\cos\varphi}$

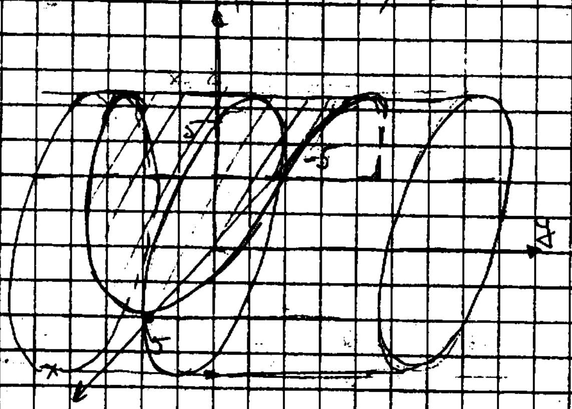
$\overline{AB}: y=-1 \Rightarrow \rho=-\dfrac{1}{\sin\varphi}$

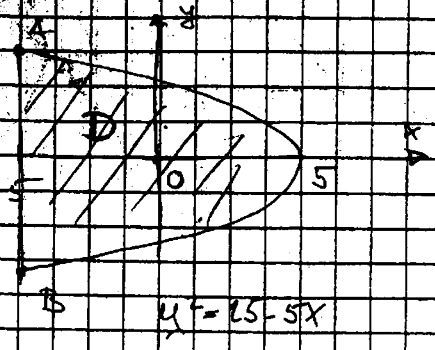
$D_1: \begin{cases} 0\le\rho\le\dfrac{1}{\cos\varphi} \\ -\pi/4\le\varphi\le\pi/4\end{cases}$ $\quad D_2: \begin{cases} 0\le\rho\le\dfrac{1}{\cos\varphi} \\ -\dfrac{3\pi}{4}\le\varphi\le-\dfrac{\pi}{4}\end{cases}$

$\boxed{24}$ 6.5.B.

$\boxed{21}$ $\Gamma: x^2+z^2=25$ $[y^2=25-5x]$

кружни цилиндар паралелан y-оси





$y^2=25-5x$

$\left.\begin{array}{l} y^2=25-5x \\ x=-5 \end{array}\right\}$ $y^2=50$, $y=\pm 5\sqrt{2}$

$A(-5,-5\sqrt{2})$ $B(-5,5\sqrt{2})$

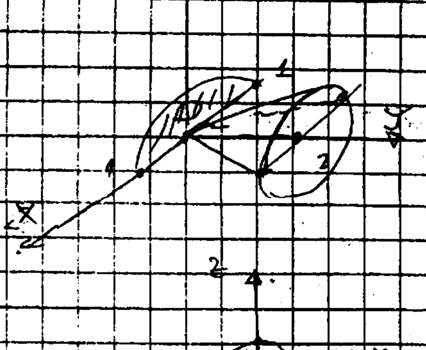
$\Gamma: z=\pm\sqrt{25-x^2}$

$z=\sqrt{25-x^2}$ $\quad p=z'_x=\frac{-x}{\sqrt{25-x^2}}$ $\quad q=z'_y=0$

$$P=2\iint_D\sqrt{1+p^2+q^2}\,dx\,dy=2\iint_D\sqrt{1+\frac{x^2}{25-x^2}}\,dx\,dy=2\iint_D\frac{5\,dx\,dy}{\sqrt{25-x^2}}$$

$$P=10\int_{-5}^{5}dx\,\frac{1}{\sqrt{25-x^2}}\int_{-\sqrt{25-5x}}^{\sqrt{25-5x}}dy=10\int_{-5}^{5}\frac{dx}{\sqrt{25-x^2}}\cdot 2\sqrt{25-x^2}=20\int_{-5}^{5}dx=20\cdot 10=200$$

$\boxed{28}$ $\Gamma: y=\sqrt{x^2+z^2}$ $\quad [z\geq 0,\ y\leq 2]$

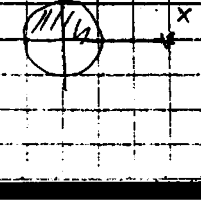


$\Gamma: y=\sqrt{x^2+z^2}$ $\quad p=y'_x=\frac{x}{\sqrt{x^2+z^2}}$ $\quad q=y'_z=\frac{z}{\sqrt{x^2+z^2}}$

$$P=\iint_D\sqrt{1+p^2+q^2}\,dx\,dz$$

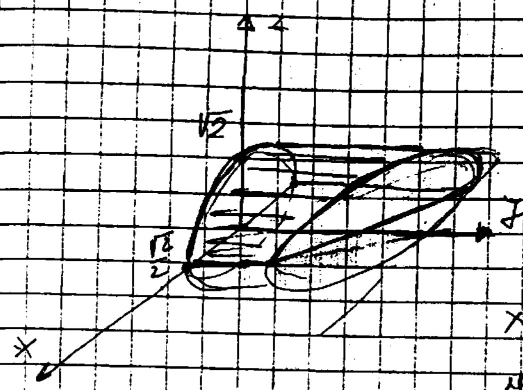
$$=\iint_D\sqrt{2}\,dx\,dz=\sqrt{2}\iint_{D^*}\rho\,d\rho\,d\varphi$$

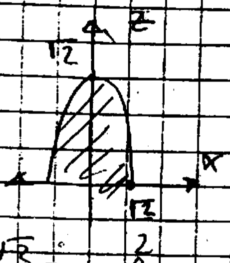
$$=\sqrt{2}\int_0^{\pi}d\varphi\int_0^2\rho\,d\rho=\frac{\sqrt{2}}{2}4\pi=2\sqrt{2}\pi$$



$D^*=\begin{cases}0\leq\rho\leq 2\\ 0\leq\varphi\leq\pi\end{cases}$

6.6.B (13) $4x^2+z^2=2,\quad z=0,\quad x+y=\sqrt{2}$





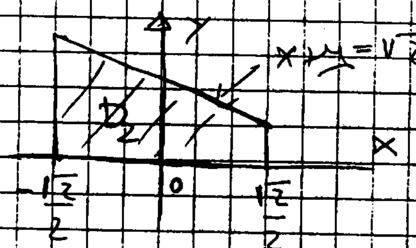
I начин: проектуємо на XoZ

$$V=\iint_{D_1}(\sqrt{2}-x)\,dx\,dz=\int_0^{\pi}d\varphi\int_0^1\left(\sqrt{2}-\frac{\sqrt{2}}{2}\rho\cos\varphi\right)\rho\,d\rho=\frac{\sqrt{2}}{2}\pi$$

$x=\frac{\sqrt{2}}{2}\rho\cos\varphi$

$z=\sqrt{2}\rho\sin\varphi$

II начин: проєк. на Xoy



$$V=\iint_{D_2}\sqrt{2-4x^2}\,dx\,dy=\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}dx\int_0^{\sqrt{2}-x}\sqrt{2-4x^2}\,dy$$

$$V=\frac{\pi}{\sqrt{2}}$$

$|y| \le x$ $\quad x-|y| \ge 0$

$x \ge |y| = \begin{cases} y, & y \ge 0 \\ -y, & y<0 \end{cases}$

$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$dx\,dy$

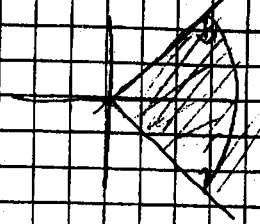
$\int_0^{\pi}\int_0^{2\pi} (\rho^2 \sin\varphi\cos\varphi)\, \rho\, \boxed{d\rho\, d\varphi} =$

$x \ge y \quad y \ge 0$

$x \ge -y \quad y<0$

$y \le x \quad y \ge 0$

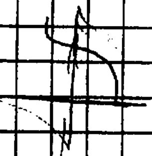
$y \ge -x \quad y<0$



$\begin{vmatrix} x'_\rho & x'_\varphi \\ y'_\rho & y'_\varphi \end{vmatrix} = \rho$

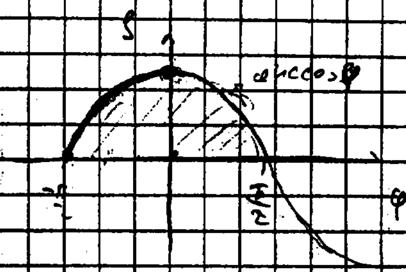
$\int_{-\pi/2}^{\pi/2}\int_0^{\cos\varphi}$

$\varphi \quad \rho$



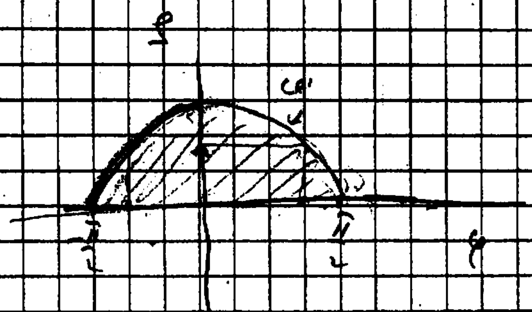
$-\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$

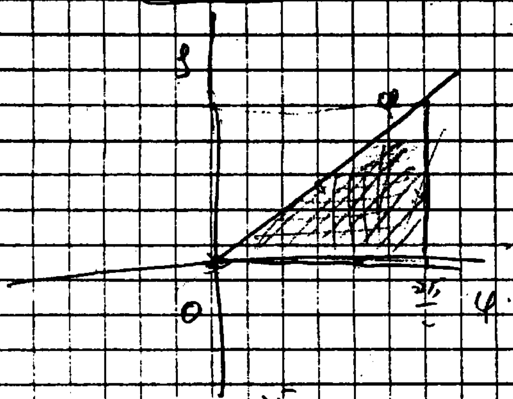
$0 \le \rho \le \cos\varphi$

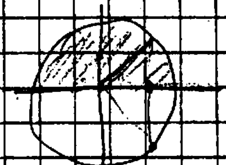


$\rho \le \cos\varphi \Rightarrow \varphi \ge \arccos\rho$

$\int_0^1 \int_{-\arccos\rho}^{\arccos\rho}$

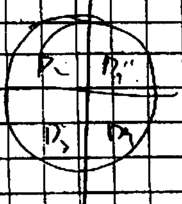






$\iint_D |xy|\,dx\,dy$

$\int_0^{2\pi}\int_\rho^{2\pi}$



$\iint_{D_1} xy\,dx\,dy + \iint_{D_2} xy\,dx\,dy \quad \int_{D_1}|xy| + \int_{D_2}|xy| \quad \int_{D_3}|xy| \quad \int_{D_4}|xy|$